M241-MATLAB (P. Staley) Project One

First Order Systems of Linear Ordinary Differential Equations With Constant Coefficients

Background Mathematics.

From the first semester calculus we know that the exponential growth initial value problem:

$y' = ky$, and $y(0) = y_0$

has the solution:

$y = y_0 e^{kt}$

An analogous ordinary differential equation (ODE) problem in two variables might be—

$y_1' = 2 y_1 - 12 y_2$

$y_2' = y_1 - 5 y_2$

with initial conditions $y_1(0) = 15$ and $y_1(0) = 4$.

For this initial value problem we postulate a solution of the form $y_i = c_i e^{kt}$ for some values of c_1, c_2 and k. Plugging this into both ODEs and dividing by e^{kt} results in the system of linear equations—

$k c_1 = 2 c_1 - 12 c_2$

$k c_2 = 1 c_1 - 5 c_2$

in matrix form we have—

$$\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = k\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Ah Hah! k must be an eigenvalue of $\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ must be a corresponding eigenvector. For this matrix MATLAB tells us that the eigenvalues are -1 and -2. They have corresponding eigenvectors $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. It turns out that sums and scalar multiples of solutions to this type of differential equation also satisfy the system. Therefore we can look for a solution to the initial value problem of the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b_1 e^{-t}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + b_2 e^{-2t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Now for t = 0 we have the initial conditions—

$$\begin{bmatrix} 15 \\ 4 \end{bmatrix} = b_1\begin{bmatrix} 4 \\ 1 \end{bmatrix} + b_2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

in matrix form this is

$$\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 4 \end{bmatrix}$$

using MATLAB's rref function on $\begin{bmatrix} 4 & 3 & 15 \\ 1 & 1 & 4 \end{bmatrix}$ gives the reduced row echelon form $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$, i.e. $b_1 = 3$ and $b_2 = 1$. So the solution to the initial value problem is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 3e^{-t}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + e^{-2t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

or in non-matrix notation:
$y_1(t) = 12\ e^{-t} + 3\ e^{-2t}$
$y_2(t) = 3\ e^{-t} + e^{-2t}$

Thus the **general technique to solving systems of first order linear differential equations with constant coefficients** is—
1. Use MATLAB's eig function to find the eigenvalues and eigenvectors of the coefficient matrix.
2. Use MATLAB's rref function to find the coefficients of the $e^{\lambda_i t}$ terms that satisfy the initial conditions.
3. Express the solution functions as linear combinations of the $e^{\lambda_i t}$ terms

---

What To Do for Project One

Part 1. Use MATLAB to solve the initial value problem:
$y_1' = -3\ y_1 + 10\ y_2$
$y_2' = 5\ y_1 + 2\ y_2$
$y_1(0) = -2$ and $y_1(0) = 7$.

Part 2. Download the P1_coeffs.txt and P1_initials.txt files from the class website (www.staley-classes.org then right click and "save target as" for the Project 1 "coeffs" and "initials" hyperlinks). Use the "load" MATLAB command to get theses matrices into your MATLAB workspace. Solve the initial value problem corresponding to that data.

For both parts turn in the following:
1. The solution in legible form.
2. A cleaned up printout of the MATLAB command window showing your work.
Answers to the following questions:
3. If all the eigenvalues are negative what does that imply for the eventual solution state, i.e. what happens to all the y_i as t goes to ∞?
4. What if all but one of the eigenvalues is negative?
5. Read about the MATLAB function exp and suggest a strategy for producing solution values for various values of t.