Leibniz and the Spell of the Continuous

Hardy Grant

The College Mathematics Journal, September 1994, Volume 25, Number 4, pp. 291-294.

Hardy Grant was born in Ottawa, Canada, and holds degrees in mathematics from Queen's University, Kingston, Ontario, and McGill University, Montreal. He has taught at York University, Toronto, since 1965. He takes a special interest in the role of mathematics in cultural history, and he has given an undergraduate humanities course on this subject for many years. His other enthusiasms include travel, birding, computer programming and pre-1950s popular music.

The famous fictional detective Philo Vance once dabbled in the history of mathematics. One of the keys to a particularly baffling murder, he told a bemused policeman, was the fact that the mathematicians of the seventeenth century, unlike their modern descendants, dealt only with well-behaved functions. "Neither Newton nor Leibniz nor Bernoulli," said the great sleuth, "ever dreamed of a continuous function without a tangent" [**4**]. Vance's legendary erudition was usually sound, and this case was no exception. In the seventeenth century such mathematical bizarreries as continuous but nowhere differentiable functions were indeed still far in the future.

The Law of Continuity

For no thinker of that age was the seeming regularity of the mathematical universe more significant than for Leibniz. This pioneer contributor to the infinitesimal calculus was also (of course) a great philosopher, whose metaphysical views were profoundly shaped by his mathematical knowledge and experience. Mathematics was for him a body of eternal truths describing an objectively known reality; moreover he felt, like many before and after him, that the clarity of its ideas and the rigor of its arguments made mathematics the paradigm of an exact and certain science. Hence he came to see it as a model for inquiry in other fields, and as a source of potential insight into God's creation and governance of the world. In particular the continuity so conspicuous in the curves and functions of contemporary mathematics underwrote for Leibniz one of the cardinal principles of all his thought. In what follows I sketch the impact of a mathematically conceived *Law of Continuity* on several diverse aspects of this protean thinker's mature philosophy.

He expressed this fundamental *lex continui* in various ways. In one informal statement he identified it with the old saying that "nature makes no leaps," adding by way of elaboration that "we pass always from the small to the great, and the reverse, through the medium" [**1**]. But his attempts to describe the Law of Continuity more rigorously have a decidedly mathematical air. He wrote in 1687:

> When the difference between two instances in a given series or that which is presupposed [*deux cas....in datis ou dans ce qui est posé*] can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought or in their results [*in quaesitis ou dans ce qui en resulte*] must of necessity also be diminished or become less than any given quantity whatever. [**2**, p. 539]

This formulation, with its curious mix of French and Latin, hints instructively at the power of good terminology. Lacking any equivalent of our "independent variable," Leibniz here lapsed into a vagueness which leaves his precise meaning open to debate.

But on its most natural interpretation the statement describes the continuity of functional dependence. Indeed, short only of the crucial stipulation that $|y_1 - y_2|$ can be made arbitrarily small by taking $|x_1 - x_2|$ *sufficiently* small, Leibniz here set forth the familiar $\varepsilon - \delta$ characterization of the continuity of a function $y = f(x)$, nearly 150 years before its rigorous enunciation by Bolzano (1817) and by Cauchy (1821).

But the rule just quoted depends, said Leibniz, on a still "more general principle," namely that "as the given quantities are ordered, so the affected quantities are ordered also [*datis ordinatis etiam quaesita sunt ordinata*]" [**2**, p. 539]. Again the utterance is cryptic, but—as his subsequent illustrations help to make clear—Leibniz here envisaged the kind of continuity in which (as we say) the limit of a convergent sequence inherits the properties of the sequence's terms. His favorite examples of such sequences drew on both mathematics and physics: a succession of regular polygons progressively filling a circle, a sequence of velocities decreasing toward zero. He recognized that in these and other cases the limit does differ in obvious ways from the sequence's terms, but he tended to brush such distinctions aside:

> Although it is not at all rigorously true that rest is a kind of motion... any more than it is true that a circle is a kind of regular polygon... [these limits of sequences] nevertheless have the same properties as if they were included in the series. [**2**, p. 887]

So far as I know, he admitted no exceptions to this striking generalization—even such apparently stark counterexamples as the sequence $\{1/n\}$, positive terms approaching a nonpositive limit. It is intriguing to remember that ancient Greek thought, in this context, declined to transfer *qualitative* properties from the terms of a sequence to the limit. Greek mathematics might seek an arbitrarily close approximation to, say, the area of a circle by way of inscribed polygons, but Greek philosophy insisted that the curvilinear and the rectilinear are fundamentally distinct. Similarly the ancients considered that even the slowest of motions is quite different, in nature and even in value, from a state of absolute rest. Leibniz' stance must on the face of it seem much less subtle, even perverse. But he had his own deep and sufficient reasons—as we shall see.

What bred in him so passionate a commitment to the continuous? Certainly the ultimate wellsprings were religious and aesthetic: The order and predictability that he saw everywhere in the nature of things, the absence of chaos and caprice, were gifts of a benevolent God, and the source of the world's perfection and beauty. But God's design and operation of the universe are (he felt) at bottom mathematical--"the sovereign wisdom, the source of all things, acts as a perfect geometrician" [**2**, p. 539]. And geometry is "but the science of the continuous" [**3**, p. 185]. Now this last declaration had been a commonplace, ritualistically repeated, since antiquity—as Leibniz well knew. But he gave it a new and characteristic twist. Traditionally, the statement had aimed merely to contrast the intuitively obvious continuity (absence of gaps or jumps) of geometrical entities like line segments with the discreteness of the objects of arithmetic, the natural numbers. But Leibniz, surveying an enormously richer stock of geometrical objects than the Greeks ever knew, expanded the old saying into a celebration of the unfailing continuity exhibited by all the curves and functions of contemporary mathematics. In particular he exulted that the inherently reasonable behavior of these objects made them amenable to the powerful new techniques of analytic geometry and the calculus. In the study of such curves, he wrote, “no single instance can be adduced of any property suddenly arising or vanishing without the possibility of our determining the intermediate transitions, the points of inflection and singular points, with which to render the change explicable” [**3**, p. 185].

Continuity in Nature

And if mathematics presents no discontinuities, neither does the world of physical experience, for indeed (said Leibniz) “the more one knows [Nature] the more geometric one finds her” [**2**, p. 541]. The ubiquity of the Law of Continuity in geometry

> soon informed me that it could not fail to apply also in physics. [For] in order for there to be any regularity and order in Nature, the physical must be constantly in harmony with the geometrical, and... the contrary would happen if wherever geometry requires some continuation physics would allow a sudden interruption. [**3**, p. 185]

On at least one occasion Leibniz flirted briefly with the kind of spectacular natural discontinuity now studied by chaos theorists: He pictured, as an example of a tiny cause with immense effects, a small spark destroying an entire city by igniting a quantity of gunpowder. But he dismissed such apparent anomalies as not really outside the general rule [**2**, p. 541].

In time this universality of the continuous acquired for him the status of a great overarching principle, one of the supreme and unchallengeable determinants of all his thought. Continuity became not a fact to be verified in each new investigation but an *assumption* made in advance and hence a source and test of other conclusions. Here then is the clue to Leibniz’ insistence, with counterexamples swept under the rug, that the properties of a convergent sequence’s terms *always* carry over to the limit: This he came to regard as a necessary consequence of a higher and surer truth. “Since we can move from polygons to a circle by a continuous change and without making a leap, it is also necessary not to make a leap in passing from the properties of polygons to those of a circle,” for “*otherwise the law of continuity would be violated*” [**2**, p. 887; emphasis added]. In the same spirit he urged that any proposed description of physical phenomena that ran counter to this omnipresent rule must be abandoned. He took pride in basing solely on this criterion, *no further argument being required*, a telling rebuttal of one of Descartes’ ventures into mechanics. Descartes had laid it down that if two bodies B and C, moving on a straight line with equal velocities, collide, then each will be reflected with the velocity of approach. But Descartes also claimed that if B’s velocity exceeds C’s, however slightly, then C will be reflected as before but B will continue in its original direction. Leibniz saw that the passage from the first of these scenarios to the second drew a large difference in outcome from a small variation of initial conditions, and so he rejected the second of Descartes’ conclusions as incompatible with the guaranteed smoothness of nature’s working [**2**, p. 540]; cf. [**3**, p. 186].

In another context the continuity in the geometry of the conic sections inspired Leibniz to one of his grandest visions. He pointed out that although ellipses and parabolas look dissimilar, a parabola can be regarded as the limit obtainable from an ellipse by letting one focus go to infinity. Obviously the passage from initial ellipse to final parabola traverses a continuum of intervening ellipses. (Kepler (1604) had viewed *all* the conic sections in just this spirit, in an important early statement of the principle of continuity.) To Leibniz the ellipse-parabola relation suggested an analog in a realm seemingly remote—biology. He could believe, he wrote, that the world's living creatures, though visually as different from one another as ellipses and parabolas, form like these a continuum. Any missing rungs in this *scala naturae,* any gaps between known species, he proclaimed confidently, will be filled as naturalists discover new forms. Points of seeming discontinuity, like the divide between plants and animals, are actually occupied by organisms sharing traits with neighbors on both sides. These pronouncements placed Leibniz in one of Europe's oldest intellectual traditions, for the idea of a *Great Chain of Being* has been set forth by many pens since Plato's time. But to this ancient theme he added an extra variation, which perhaps only a mathematician would voice: "When the essential determinations of one being approximate those of another, all the properties of the former should also gradually approximate those of the latter" [**3**, pp. 186-188]. That is, each biological character is a continuous function of position on the ladder of living things. No one before Leibniz, and no one after him, ever conceived the Great Chain of Being in such specifically mathematical terms.

The assumption of all-pervasive continuity colored the great philosopher's widest perspectives on the cosmic order. The Law of Continuity, applied to the temporal sequence of the world's events, entails that every physical occurrence can and must be explained in terms of preceding states. And just as "there is a perfect continuity reigning in the order of successive things, so there is a similar order" in the simultaneous; the great law holds sway in space as in time [**3**, p. 186]. Continuity underwrites the organicism so dominant in Leibniz' world view. Atomic theories of matter, which postulate disjoint particles in otherwise empty space, cannot be valid, for they would allow discontinuities in nature's operations [**3**, pp. 188-189]. (Note again the status of continuity as axiomatic in the sense of unchallengeably true, and as a basis for vital deductions.) The universe, then, is a "plenum," full everywhere, a spatial continuum. Each of its parts affects and is affected by each of the others, in mutual accommodation and influence—the "pre-established harmony" conferred by God on this best of all possible worlds.

Obviously one cannot ascribe such conclusions wholly to Leibniz' experience of mathematics. But that science, in his eyes the *modus operandi* of God's creativity, often served him as a guide to the shapes and limits of cosmic arrangements. How then might his world view have come to terms with the later realization that freakish objects and quirky behavior occur even in the apparently orderly realm of mathematics? By what adjustment might his metaphysics reflect (say) the discovery of functions discontinuous everywhere, or the insight that the sum of a convergent series of continuous functions need not be continuous? Of course we cannot know. But as it stands, his philosophy seems to echo an age of comparative mathematical innocence, when all the curves and functions under study exhibited the reassuring regularity of the continuous.

References

1. Leibniz, *New Essays Concerning Human Understanding*, 3rd ed., tr. Alfred Gideon Langley, Open Court, La Salle, IL, 1949, p. 50.
2. Leibniz, *Philosophical Papers and Letters*, tr. and ed. Leroy E. Loemker, University of Chicago Press, 1956.
3. Leibniz, *Selections*, Philip P. Wiener, ed., Scribner's, New York, 1951.
4. S. S. Van Dine, *The Bishop Murder Case*, Scribner's, New York, 1928, Chapter 21.