Sines & Cosines of the Times
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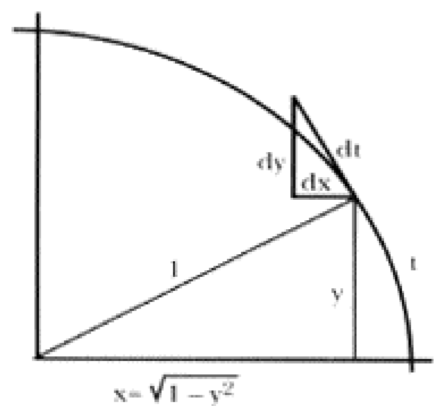
Why does the derivative of the sine equal the cosine? Or the derivative of the tangent equal the square of the secant? One answer, that you learned early in your calculus course, is that these rules can be proved. In fact, your instructor probably proved the first from the definition of derivative, having first convinced you that

$$\lim_{x\to 0} (\sin x)/x = 1,$$

and proved the second by using the quotient rule. But, after all, the trigonometric functions are defined geometrically; one ought to be able to understand their derivatives geometrically as well. If we look back at the history of these functions and their relationship to the history of calculus, we can do exactly that.

Today, we generally consider the sine and the other trigonometric functions as numerical functions of real numbers, where the numbers in the domain can be thought of as measures of angles. But until the time of Euler in the mid-eighteenth century, sines were certain lines in circles of a given radius, the lines being generally associated not to angles but to arcs. (Hence the term arcsine = arc of a given sine for the inverse function of a sine.) Thus, in the figure, we have $y = \sin t$, where t is the measure of the arc and, for convenience, we take the radius equal to 1; similarly, $x = \cos t$.

So suppose we want to take the derivative of the sine function with respect to the arc. In the eighteenth century, this meant that we would find the ratio of the infinitesimal change of $y = \sin t$ to the infinitesimal change in t. Now an infinitesimal change in the arc t can best be represented by drawing an infinitesimal tangent line to the circle at the end of the arc labeled t. If we consider this tangent as the hypotenuse of a right triangle, then the vertical leg represents dy, the infinitesimal change in the sine. Since the infinitesimal triangle is similar to the original large triangle, the laws of similarity show that $dy/dt = x/1$, or $d(\sin t)/dt = \cos t$, as desired. (Exercise: Develop an analogous argument to find the derivative of the tangent.)



The geometrical arguments giving the derivatives of the sine and tangent first appeared in print in a posthumously published paper of Roger Cotes (1682–1716), the editor of the second edition of Newton's *Principia*. The first appearance of the sine argument in a calculus text, however, was in *A New Treatise of Fluxions* (1737) by Thomas Simpson (1710–1761), famous today mostly for Simpson's rule for numerical integration by parabolic approximation. Simpson was one of a group of private teachers in England who met the growing demand of the English middle class for mathematical knowledge. The textbook grew out of Simpson's membership in the Mathematical Society at Spitalfields, whose rules made it the duty of every member "if he be asked any mathematical or philosophical question by another member, to instruct him in the plainest and easiest manner he is able."

Interestingly, although neither Newton nor Leibniz considered explicitly the derivatives of the trigonometric functions, they did deal with their power series and their differential equations. For example, Leibniz used the same figure with its differential triangle to conclude that $dy^2 + dx^2 = dt^2$, or, since $x = \sqrt{1 - y^2}$ and $dx = y\,dy/\sqrt{1 - y^2}$, that $dy^2 + y^2dt^2 = dt^2$. Assuming, then, that dt is a constant increment and therefore that its differential is 0, Leibniz applied his differential operator d to both sides of the equation to get $d(dy^2 + y^2dt^2) = 0$. Using the product rule for differentials on the left side, he simplified this into $2dy(d\,dy) + 2ydy\,dt^2 = 0$ or $d^2y + ydt^2 = 0$ or, finally, into the familiar differential equation of $y = \sin t$: $d^2y/dt^2 = -y$. Note that Leibniz's method of manipulating with second order differentials explains our seemingly strange placement of the 2's in the modern notation for second derivatives.

Although arguments using infinitesimals were replaced by arguments using limits in the early nineteenth century, their heuristic value in the learning of calculus remains. And in recent decades, the work of Abraham Robinson on non-standard analysis has shown that these arguments can even be modified to meet modern standards of rigor. Rigorous arguments notwithstanding, the history of many of the concepts of calculus helps us to develop an intuitive understanding of the basic ideas of the subject, an understanding necessary for us to apply these techniques to the solving of problems.