Graphs of Rational Functions for Computer Assisted Calculus

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In this capsule, we suggest some calculus problems whose solutions involve pencil-and-paper techniques and some form of computer assistance. These are problems that can be used as calculus laboratory projects. We expect the computer to act as a strong and convenient number-cruncher, but we expect the student to supply the conceptual framework. For some of the problems below, finding the proper scaling so that one can see the extrema is a bit difficult, but we feel that a student will profit from this trial-and-error experience. The main computational difficulty of these problems is approximating all the roots of a polynomial, so your computer package should have a reliable polynomial root finder.

Assuming a, b, c, and d are positive real numbers, we determine the important properties of the graphs of the family of rational functions,

$$f(x) = \frac{ax^2 + b}{(x + b)^2} + \frac{cx^2 + d}{(x + d)^2}. \tag{1}$$

(We encountered this family in a problem in the *SIAM Review* [**3**], where f is described as a mean-squared-error function for a class of regression models, and the author asks for conditions under which the minimum of f on $[0, \infty)$ is unique.) As we will show in the following discussion, this family of functions is a good place to make the transition from usual textbook rational function graphs to graphs that should be analyzed with the aid of a computer. (Students who hope to push buttons and get sufficient information will be disappointed.)

In the following list we suggest some problems, following each with remarks about its solutions. Problems 1 and 2 should be solved via pencil and paper analysis, while Problems 3 and 4 should be solved with the aid of one or two of the many available computer programs. (We have used *Derive* and *Mathematica*, but these powerful programs are not necessary.)

Problem 1. Graph the function

$$f(x) = \frac{ax^2 + b}{(x + b)^2}, \tag{2}$$

finding the asymptotes, monotonicity intervals, concavity intervals, extrema, and inflection points. (The assumption that a and b are positive insures that each graph has the same basic shape.)

We prefer that students graph some specific instances of these functions using hand calculations and then graph the general case, labeling important points on the graph with expressions involving a and b. The derivatives are messy, but we have prepared our students by doing similar calculations before looking at these problems. They should especially use hand calculations to find the minimum at $x = 1/a$, the unique inflection point at $x = 3/(2a) + b/2$ and the horizontal asymptote at $y = a$, since it would be possible to overlook these features on a computer-generated plot.

Problem 2. Use the results from problem 1 to make a rough sketch of the graph of (1) for $(a, b, c, d) = (2, 3, 3, 1)$ and those values of x *not* between $x = m = \min\{1/a, 1/c\}$ and $x = n = \max\{3/(2a) + b/2, 3/(2c) + d/2\}$. Explain why it is difficult to sketch this function on the omitted domain (m, n).

As in problem 1, pencil and paper analysis is sufficient to work problem 2. By thinking of the function in (1) as the sum of two terms (each in the form of (2)), and realizing that both f' and f'' are the sum of the derivatives of the two terms, one has no difficulty sketching this function outside the interval (m, n). Although functions in the form of (1) clearly must have at least one minimum on (m, n), a complete analytical determination of the roots of the derivatives is difficult.

Problem 3. Use a function plotter to plot some examples of (2) such as $(a, b) = (2, 3)$ and $(a, b) = (3, 1)$. A student using a function plotter will notice the scaling problem that often occurs when a computer plots a function (see [**2**]). If one scales the y-axis to see more of the vertical asymptote at $x = -b$, then one will have trouble seeing the unique minimum at $x = 1/a$ and vice versa.

Problem 4. Use a function plotter program to plot (1) for $(a, b, c, d) = (2, 3, 3, 1)$. Use the zoom technique and a root finding program to find the approximate location of the extrema in the interval (m, n) and the minimum in the interval between $-b$ and $-d$. Repeat this problem for $(a, b, c, d) = (2, 1, 2, 2)$ and $(a, b, c, d) = (0.02, 0.7, 10.0, 0.01)$.

Students will find that a scaling of the axis to show the gross features of the graph will not show the extrema in the interval (m, n). Still, they can use the zoom technique and a numerical root extraction routine to find very good approximations to the x coordinates of the extrema. (Using a computer algebra system to extract the roots of the fourth degree polynomial in the first derivative's numerator would be slow and its results difficult to interpret.) After working the first two examples of problem 4, students might guess that the graphs of functions of the form (1) all have the same shape. The last example illustrates that this guess is incorrect. For it, students should find four extrema and three inflection points. Figure 1 and Figure 2 are plots of this example with a different scaling on each plot.

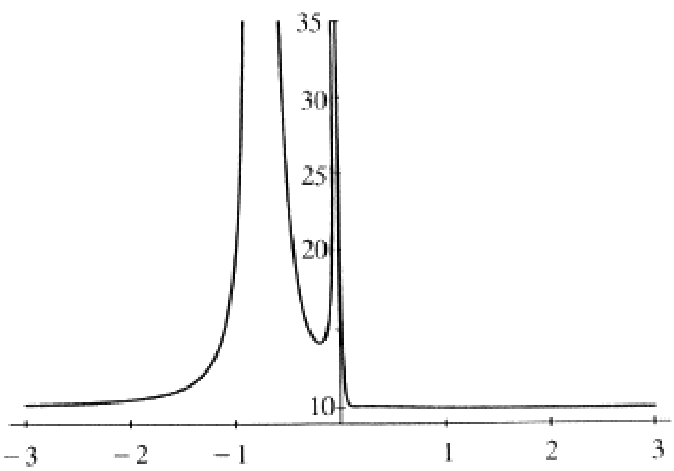

Figure 1

Plot of (1) with $(a, b, c, d) = (0.02, 0.7, 10.0, 0.01)$ showing a minimum and two vertical asymptotes.

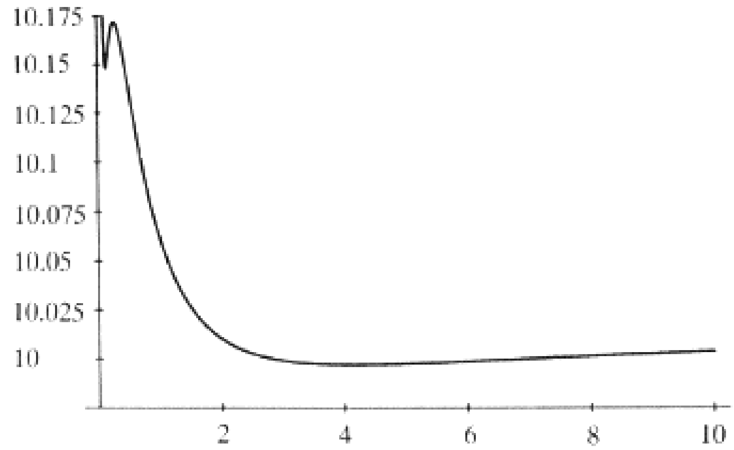

Figure 2

Plot of (1) with $(a, b, c, d) = (0.02, 0.7, 10.0, 0.01)$ showing three extrema.

The instructor might wish to discuss the implications of the facts that the graph of (1) must have positive concavity for x less than m and that the degree of the numerator of the derivative of (1) is only four.

We conclude by noting the existence of a computational test to determine if the functions (1) (with b not equal to d) have one or two minima in the interval (m, n) (see [**3**]). One can determine if the roots of a fourth degree polynomial (such as the numerator of the derivative of (1)) are real, complex, or repeated by computing the discriminant of the polynomial directly from its coefficients and applying results from the theory of equations (see [**1**]). Using a few facts about any function in the form of (1) with the positivity assumptions on a, b, c, and d, it can be shown that if the discriminant is zero or negative then the graph of function (1) has one minimum in the interval (m, n), while if the discriminant is positive then the graph has two minima and a maximum in (m, n).

References

[**1**] W. S. Burnside, *The Theory of Equations*, Dover Publications, New York, 1960.

[**2**] F. Demana and B. Waits, Pitfalls in graphical computation, or why a single graph isn't enough, *The College Mathematics Journal*, 19(1988) 177–183.

[**3**] T. S,. Lee, On the uniqueness of a minimum problem, *SIAM Review*, 28(1986) 395.