How Not to Land at Lake Tahoe!

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The following problem gives a simplified model of landing an airplane. It is adapted and extended from Trim [**1**] and is regularly presented in first semester calculus at my campus, where it is unanimously enjoyed and wins some converts to the methods of calculus.

Problem. An aircraft landing approach pattern is shaped generally as in Figure 1 below. The following conditions are imposed:

a) The cruising altitude is h when descent begins at a horizontal distance L from the airstrip.

b) A constant horizontal airspeed U must be maintained throughout descent (somewhat unrealistic).

c) At no time must the vertical component of acceleration exceed (in absolute value) some fixed constant k, $0 \leq k \ll g$, where g is the acceleration constant for gravity; i.e., $g = 32\ \mathrm{ft/sec^2}$ (English units).

Model the plane's approach path by means of a cubic polynomial, using a coordinate system with origin at the beginning of the runway, so that descent starts at the point $(x, y) = (-L, h)$, in units of your choice. Impose suitable conditions at the beginning of descent and at touchdown. Discuss the implications of condition c) above, in the cases: 1) transcontinental flight; and, 2) peculiar airport situations (such as at South Lake Tahoe, CA).

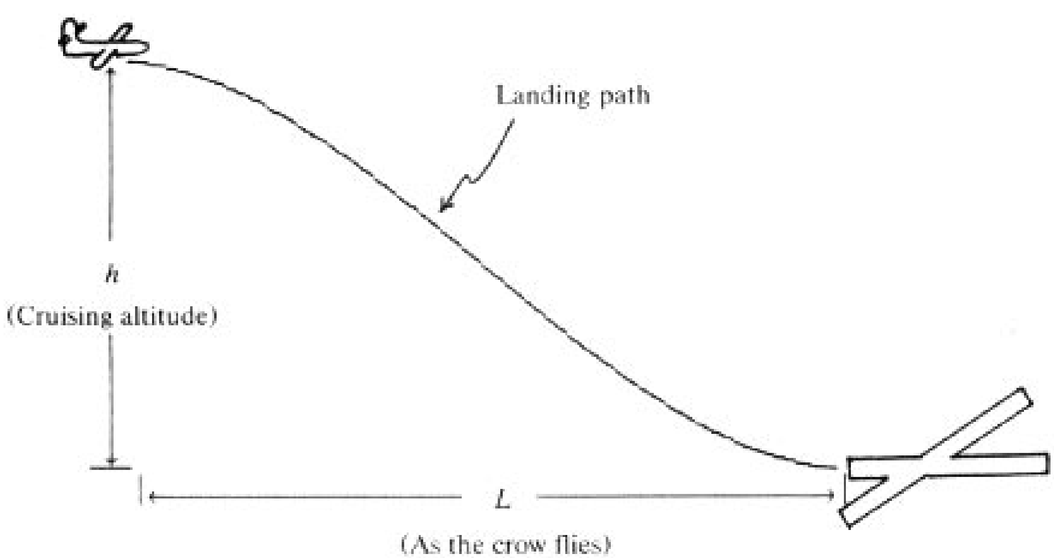

Figure 1

Solution.

We let the landing path have the form:

$$y(x) = ax^3 + bx^2 + cx + d.$$

The following reasonable conditions are imposed:

$$\left.\begin{aligned} &y(0) = 0 \quad \text{(touchdown)} \\ &\left.\frac{dy}{dx}\right|_{x=0} = 0 \text{ (no crash)} \end{aligned}\right\} \text{ imply } c = d = 0;$$

$$\left.\begin{aligned} &y(-L) = h \quad \text{(descent)} \\ &\left.\frac{dy}{dx}\right|_{x=-L} = 0 \text{ (no dive)} \end{aligned}\right\} \text{ imply } \begin{aligned} a &= 2h/L^3 \\ b &= 3h/L^2. \end{aligned}$$

Thus these conditions give:

$$y(x) = h\{2(x/L)^3 + 3(x/L)^2\},$$

where x/L is a dimensionless coordinate.

By using the chain rule (with the simplification of constant horizontal airspeed component $dx/dt = U$), we obtain:

$$v_y = \frac{dy}{dt} = \frac{6Uh}{L}\{(x/L)^2 + (x/L)\}$$

and

$$a_y = \frac{d^2y}{dt^2} = \frac{6U^2h}{L^2}\{2(x/L) + 1\}.$$

Now,

$$(a_y)_{\max(\min)} = (\substack{+\\-})\frac{6U^2h}{L^2},$$

which occur at $(0, 0)$ and $(-L, h)$, respectively. (Hence the airport approach resembles a ride in an elevator, where we "feel" the motion only at the top and bottom of descent.) Since we want $|a_y|_{\max} \le k \ll \mathrm{g}$, we have:

$$\frac{6U^2h}{L^2} \le k.$$

Implications. 1) Los Angeles to New York (LAX to JFK) transcontinental flight aboard a jumbo ("heavy'') jet.

$$L \geq \sqrt{\frac{6U^2h}{k}}$$

If U and h are large, while k is small, L (the distance from the airport where descent begins) must be relatively big. On such a flight, with an airspeed of $U = 600$ mph and a cruising altitude of $h = 37{,}000$ ft, the author discovered, from his own experience, that descent began at his home near Scranton, PA, about 130 miles from New York! This will make the value of k, which is given by:

$$k = \frac{6U^2h}{L^2 \cdot (3600)^2},$$

come out to $k = 0.36$ ft/sec^2. [The value $(3600)^2$ converts k from ft/hr^2 to ft/sec^2, since a mix of units such as mph and ft is actually in use by airlines (as opposed to mathematicians?)!]

2) San Francisco to South Lake Tahoe. Here we solve for U and obtain:

$$U \leq \sqrt{\frac{kL^2}{6h}}.$$

If L and k are small but h is relatively large, and if we don't want our coffee or the flight attendant to go floating about the cabin, then the airspeed must be kept low.

A few years ago the author had occasion to visit his two sisters-in-law (who are both in applied mathematics, dealing blackjack in the casinos) at Lake Tahoe. As our "gambler's special'' aircraft crossed the last peak of the Sierra Nevada mountains ($h = 11{,}000$ ft), there was the airport, seemingly directly below us ($L = 20$ mi), and we almost dove into a landing (see Figure 2)!

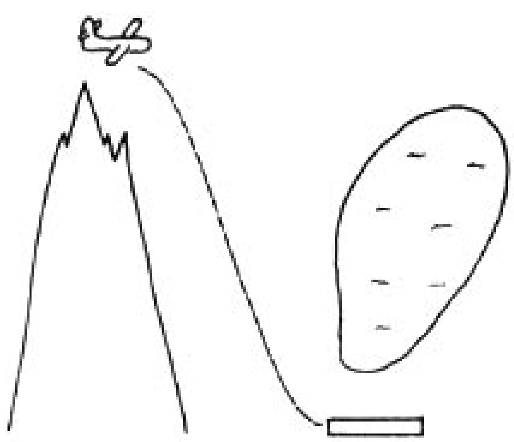

Figure 2

Our plane was, in fact, a two engine prop plane with an airspeed of about $U = 175$ mph. With the above values for U, L, and h, $k = 0.39$ ft/sec^2, not much different from the value of k for the transcontinental flight discussed above!

Parenthetically, because of noise restrictions aircraft are not allowed to land from over the lake to the north of the airport, and, consequently, jets cannot land at the airport at Lake Tahoe.

(Actually, I fudged a bit on the values for L and h in the example above, for the descent was somewhat more harrowing than I made it out to be. So therein lies a research project for the calculus class: to write letters and contact flight engineers at TWA and Golden West Airways for more accurate values of L, h, and U for the flights discussed.)

In practice, aircraft decrease their airspeed when landing and often engage in a banked loop around the airport in order to slow down further before touchdown. Nevertheless, the above simplistic model for the approach pattern qualitatively agrees with actual flying experience.

References

[**1**] D. W. Trim, *Calculus and Analytic Geometry*, Addison-Wesley Publishing Company, Reading, MA, 1983, p. 124.