Looking at $\sum_{k=1}^{n} k$ *and* $\sum_{k=1}^{n} k^2$ *Geometrically*

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Two commonly taught algebraic sums are

$$1 + 2 + 3 + 4 + \cdots + (n-1) + n = \frac{n(n+1)}{2} \text{ and}$$

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

The first equation has a short algebraic proof, and the second has a more intricate one. An alternative approach is to evaluate the sums geometrically. A method for each sum is presented here. Evaluating the first sum involves positioning squares and then determining area. Evaluating the second sum involves arranging cubes then determining volume. Although this article can use only two-dimensional pictures to demonstrate the second sum, manipulating (three-dimensional) physical cubes adds considerable clarity to a presentation.

To find the first sum, we position the squares with sides of length 1, as shown in **figure 1,** laying one square on the first row, two squares on the second row, three squares on the third row, and so on up to the nth row, on which we lay n squares. Thus the area of this shape equals the sum

$$1 + 2 + 3 + 4 + \cdots + (n-1) + n.$$

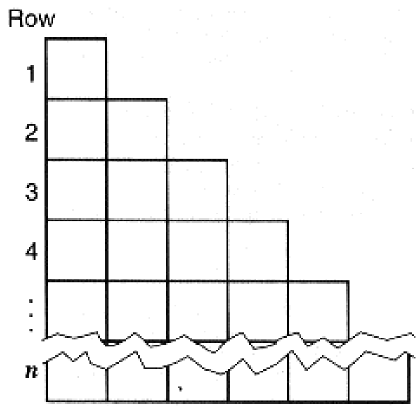


Fig. 1.

A stairstep pattern of single squares

This area can also be found if we divide the shape as shown in **figure 2.** The area of the large triangle is $n^2/2$, since the shape is made of n rows with the nth row being n units across. The remaining halved squares each have area $1/2$. Because n rows are

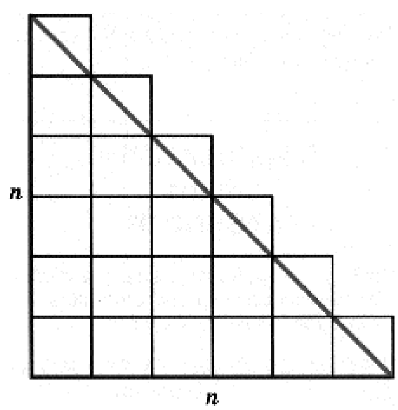


Fig. 2.

Find the area of the region using the formula for the area of a triangle: $(n)(n)/2 + n/2$.

involved, n of these triangles result, their area being $n(1/2)$. Therefore the total area,

$$\sum_{k=1}^{n} k,$$

is

$$\frac{n^2}{2} + \frac{n}{2},$$

which equals

$$\frac{n(n+1)}{2}.$$

In evaluating

$$\sum_{k=1}^{n} k$$

the area of the kth row of the figure is k. Thus if we represent

$$\sum_{k=1}^{n} k^2$$

by a geometrical figure, each row in the figure must represent the value k^2. So instead of each row being squares arranged in a line, each row needs to be cubes arranged on a square.

We create this shape, pictured in **figure 3,** by stacking layers of cubes whose edges are of length 1. Place one cube in the first layer, four cubes in the second layer, nine cubes in the third layer, and so on, down to the nth layer, in which we place n^2 cubes. By counting the number of cubes, the volume of this stack is the sum

$$1^2 + 2^2 + 3^2 + 4^2 + \cdot\cdot\cdot + (n-1)^2 + n^2.$$

To derive the equation given for this sum, we must find the stack's volume by another method. The method that worked in the first example was to separate a large triangle from the shape, calculate its area, and then find the area of the other pieces. Perhaps dividing the stack in an analogous manner will work in this example. Yet no shape is definitively analogous to the triangle. Observe, however, that a side view of the stack in **figure 3** looks exactly like **figure 2.** This view suggests that the shape we should

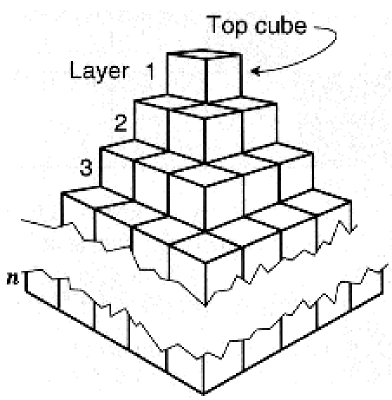


Fig. 3.

Layers of cubes

remove might be one bounded by the triangle in **figure 2,** when we view the shape from the side of the stack. In fact, the shape we will divide from the stack does have this property. It is the oblique square pyramid shown in **figure 4.** The back edge $\overline{DE}$ is perpendicular to the pyramid's square base, $ABCD$, whose sides have the same length, b, as $\overline{DE}$. Using the formula for the volume of a pyramid, the volume of the shape is (area of base)(height)$/3 = b^3/3$.

Before removing the shape from the stack, however, we must see how to slice it from a simple, solid cube. This explanation will aid in calculating the volume of the pieces in

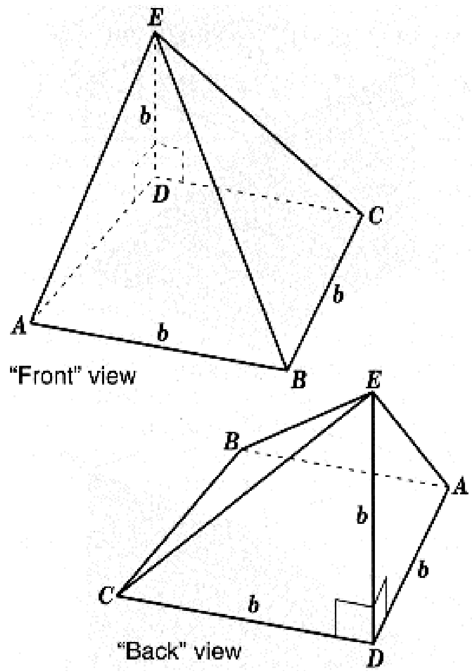
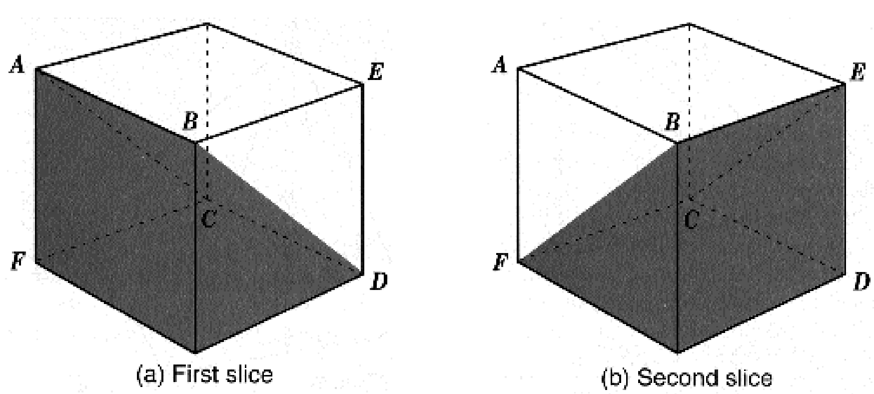


Fig. 4.

An oblique square pyramid

the stack that are not part of the pyramid. To divide the pyramid from a cube of edge *b*, and later from our stack, we need to make two slices. See **figure 5**. First divide the cube in half by cutting it from edge $\overline{AB}$ to edge $\overline{CD}$. Then, without rotating the cube, halve the cube again, slicing from edge $\overline{BE}$ to edge $\overline{FC}$. The remaining part, in front, encompassing the space common to both shaded regions is similar to the pyramid in **figure 4**. One can see that this assertion is true by considering the boundaries of the slicing and noticing that plane *ABCD* intersects plane *BECF* along the line *BC*. Since the volume of the pyramid removed is $b^3/3$, the volume of the rest of the cube is $2b^3/3$.



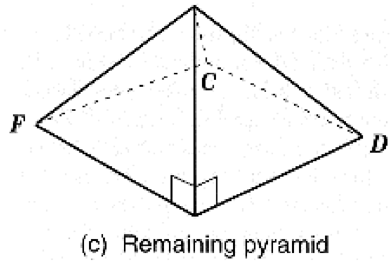


Fig. 5.
Slicing a pyramid from a cube

At last, divide the stack (from **fig. 3**), making two slices as done with the cube. See **figure 6.** The volume of the large pyramid, under the slices, is $n^3/3$, since the stack is n layers high with an n-by-n base. But what about the rest of the pieces? Notice that the only cubes in the stack that were sliced were the ones whose tops are exposed. We can see them all from the overhead view in **figure 7**. The cubes affected by only one of the slices are all divided in half, since each was cut in the same manner as the cube that was cut with the first slice in **figure 5a.** Considering those chopped *only* by the first slice, we find one on the second layer, two on the third layer, three on the fourth layer, and, in the general case, $n - 1$ cubes halved on the nth layer. Because the bottom half of these cubes are part of the large pyramid, only the top half of them remain unaccounted. Their volume is

$$\tfrac{1}{2}(1 + 2 + 3 + \cdots + n - 1).$$

From the formula for the first sum, this amount is $(1/2)\,((n - 1)n/2)$. Since the same number of half cubes remain after the second slice, the total volume of the half cubes is simply $(n - 1)n/2$. The only pieces now unaccounted for are the cubes that were cut twice. Because each cube was cut twice diagonally from edge to edge in the same way as the cube in **figure 5,** a pyramid similar to the one in **figure 5c** was sliced from each. Since these pyramids were taken into account when we found the volume of the large pyramid, two-thirds of the volume of each cube remains. The stack has n layers, so n of these twice-sliced cubes result (see **fig. 7** again), and the volume of those not in the big pyramid is $2n/3$. Therefore, the total volume of the stack, and consequently

$$\sum_{k=1}^{n} k^2,$$

equals

$$\frac{n^3}{3} \qquad + \qquad \frac{(n-1)(n)}{2} \qquad + \qquad \frac{2n}{3}.$$

(big pyramid) + (top of the halved cubes) + (remainder of the twice-sliced cubes)

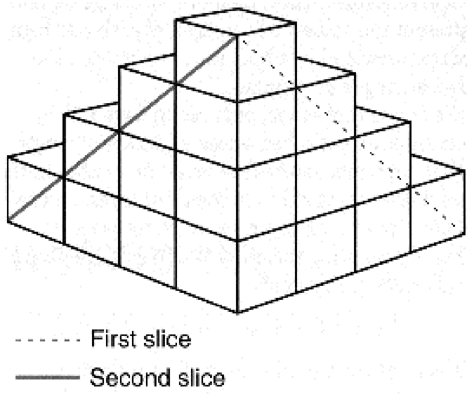


Fig. 6.
Slicing the stack

With some algebra, this expression becomes

$$\frac{2n^3 + 3(n-1)n + 4n}{6} = \frac{n(2n^2 + 3(n-1) + 4)}{6}$$
$$= \frac{n(2n^2 + 3n + 1)}{6}$$
$$= \frac{n(n+1)(2n+1)}{6},$$

the answer we expect.

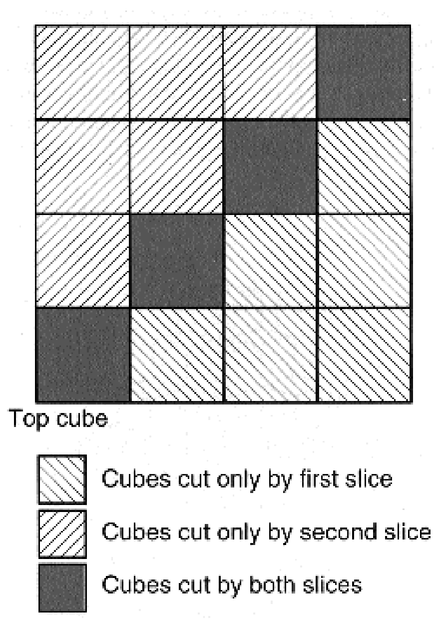


Fig. 7.
An overhead view of the stack

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