Unexpected Occurrences of the Number e

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Martin Gardner once observed [**2**, p. 34] that of the three most famous irrational numbers, π, the golden ratio, and e, the third is least familiar to students early in their study of mathematics. The number e, named by Leonhard Euler (1707–1783), is usually encountered for the first time during the second course in calculus, either through the equation

$$\int_1^e \frac{dt}{t} = 1,$$

introduced in connection with natural logarithms, or else through the equation

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

However, neither of these definitions of e provides immediate insight into this important number. As a result, students rarely come away with a solid grasp of this number beyond, perhaps, rote memory of the phrase, "It is the base of the natural logarithms." Given the importance of e, students' first meeting with this number should be a more memorable experience than the standard introductions provide.

In this note we present an expository catalog of occurrences of e in probability which might be used to increase students' appreciation of this number. Most likely, some, but not all, of these examples will be familiar to teachers of mathematics.

Example 1. Each of two people is given a shuffled deck of playing cards. Simultaneously they expose their first cards. If these cards do not match (for example, two "four of clubs" would be considered a "match"), they proceed to expose their second cards and so forth through the decks. What is the probability of getting through the decks without a single match? In [**4**, p. 281] it is shown that the answer is given by the sum,

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{52!},$$

which is the initial portion of a series for $1/e$ (based on the Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \Big).$$

This problem has appeared in the literature under the name "Hat-Check Problem," wherein the following question is asked: If n men have their hats randomly returned, what is the probability that none of the men winds up with his own hat?

Example 2. The following appeared as a Putnam examination problem [**1**]: If numbers are randomly selected from the interval $[0, 1]$, what is the expected number of selections necessary until the sum of the chosen numbers first exceeds 1? The answer is e. An elementary proof of this is given in [**6**].

Example 3. The "Secretary Problem" concerns an employer who is about to interview n applicants for a secretarial position. At the end of each interview he must decide whether or not this is the applicant he wishes to hire. Should he pass over an interviewee, this person cannot be hired thereafter. If he gets to the last applicant, this person gets the job by default. The goal is to maximize the probability that the person hired is the one most qualified. His strategy will be to decide upon a number $k < n$, to interview the first k applicants, and then to continue interviewing until an applicant more qualified than each of those first k is found. As seen in [**3**], the probability of hiring the most qualified applicant is greatest when k/n is approximately $1/e$. Moreover, this number, $1/e$, is in fact the approximate maximum probability. For example, if there are $n = 30$ applicants, the employer should interview 11 (which is approximately $30/e$) and then select the first thereafter who is more qualified than all of the first eleven. The probability of obtaining the most qualified applicant is approximately $1/e$.

Example 4. With each purchase, a certain fast-food restaurant chain gives away a coin with a picture of a state capitol on it. The object is to collect the entire set of 50 coins. Question: After 50 purchases, what fraction of the set of 50 coins would one expect to have accumulated? In [**5**] it is shown that this fraction is $1 - \left(1 - \frac{1}{50}\right)^{50}$, which is approximately $1 - 1/e$.

Example 5. A sequence of numbers, $x_1, x_2, x_3,\ldots$, is generated randomly from the interval $[0, 1]$. The process is continued as long as the sequence is monotonically increasing or monotonically decreasing. What is the expected length of the monotonic sequence? For example, for the sequence beginning .91, .7896, .20132, .41, the length of the monotonic sequence is three. For the sequence beginning .134, .15, .3546, .75, .895, .276, the length of the monotonic sequence is five.

The probability that the length L of the monotonic sequence is greater than k is given by

$$P(L > k) = P(x_1 < x_2 < \cdots < x_{k+1}) + P(x_1 > x_2 > \cdots > x_{k+1})$$

$$= \frac{1}{(k + 1)!} + \frac{1}{(k + 1)!}$$

$$= \frac{2}{(k + 1)!}.$$

If we denote $P(L = k)$ by p_k, the expected length of the monotonic sequence is

$$E(L) = 2p_2 + 3p_3 + 4p_4 + 5p_5 + \cdots.$$

We rewrite this in the form

$$
\begin{aligned}
E(L) = \quad & p_2 + p_3 + p_4 + p_5 + p_6 + \cdots \\
+ \ & p_2 + p_3 + p_4 + p_5 + p_6 + \cdots \\
+ \ & \quad\quad p_3 + p_4 + p_5 + p_6 + \cdots \\
+ \ & \quad\quad\quad\quad p_4 + p_5 + p_6 + \cdots \\
+ \ & \quad\quad\quad\quad\quad\quad p_5 + p_6 + \cdots.
\end{aligned}
$$

Adding this triangular array row by row, we obtain

$$E(L) = 1 + 1 + P(L > 2) + P(L > 3) + P(L > 4) + \cdots$$

$$= 1 + 1 + 2\left(\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots\right)$$

$$= 1 + 1 + 2\left(e - \frac{5}{2}\right)$$

$$= 2e - 3 \approx 2.4366.$$

Example 6. A slight revision of the previous example gives a more pleasing answer. First, we require the sequence to be monotonically increasing and, second, in computing the length of the sequence we include the first number that reverses the increasing direction of the sequence. Hence, the sequence beginning .154, .3245, .58, .432 is assigned a score of four and the sequence beginning .6754, .239 is assigned a score of two. Using an argument similar to that of the preceeding example, it can be shown that the expected score is that ubiquitous and fascinating number, e.

References

1. L.E. Bush, The William Lowell Putnam Mathematical Competition, *Amer. Math. Monthly* 68 (1961), 18–33.

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3. J. P. Gilbert and F. Mosteller, Recognizing the maximum of a sequence, *Amer. Statist. Assoc. J.* 61 (1966), 35–73.

4. A. P. Hillman, G. L. Alexanderson and R. M. Grassl, *Discrete and Combinatorial Mathematics*, Dellen Publishing, San Francisco, 1986.

5. R. C. Shiflett and H. S. Shultz, Can I expect a full set?, *Math. Gaz.* 64 (1980), 262–266.

6. H.S. Shultz, An expected value problem, *Two-Year College Math. J.* 10 (1979), 179.