1.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, use a graphing utility to graph the function and visually estimate the limits.

1. $h(x) = -x^2 + 4x$
   (a) $\lim_{x \to 4} h(x)$
   (b) $\lim_{x \to -1} h(x)$

2. $g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$
   (a) $\lim_{x \to 4} g(x)$
   (b) $\lim_{x \to 0} g(x)$

3. $f(x) = x \cos x$
   (a) $\lim_{x \to 0} f(x)$
   (b) $\lim_{x \to \pi/3} f(x)$

4. $f(t) = t|t - 4|$
   (a) $\lim_{t \to 4} f(t)$
   (b) $\lim_{t \to -1} f(t)$

In Exercises 5–22, find the limit.

5. $\lim_{x \to 2} x^3$
6. $\lim_{x \to -2} x^4$
7. $\lim_{x \to 0} (2x - 1)$
8. $\lim_{x \to -3} (3x + 2)$
9. $\lim_{x \to -3} (x^2 + 3x)$
10. $\lim_{x \to 1} (-x^2 + 1)$
11. $\lim_{x \to -3} (2x^2 + 4x + 1)$
12. $\lim_{x \to 1} (3x^3 - 2x^2 + 4)$
13. $\lim_{x \to 3} \sqrt{x + 1}$
14. $\lim_{x \to 4} \sqrt[3]{x + 4}$
15. $\lim_{x \to -4} (x + 3)^2$
16. $\lim_{x \to 0} (2x - 1)^3$
17. $\lim_{x \to 2} \dfrac{1}{x}$
18. $\lim_{x \to -3} \dfrac{2}{x + 2}$
19. $\lim_{x \to 1} \dfrac{x}{x^2 + 4}$
20. $\lim_{x \to 1} \dfrac{2x - 3}{x + 5}$
21. $\lim_{x \to 7} \dfrac{3x}{\sqrt{x + 2}}$
22. $\lim_{x \to 2} \dfrac{\sqrt{x + 2}}{x - 4}$

In Exercises 23–26, find the limits.

23. $f(x) = 5 - x,\ g(x) = x^3$
   (a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 4} g(x)$ (c) $\lim_{x \to 1} g(f(x))$

24. $f(x) = x + 7,\ g(x) = x^2$
   (a) $\lim_{x \to -3} f(x)$ (b) $\lim_{x \to 4} g(x)$ (c) $\lim_{x \to -3} g(f(x))$

25. $f(x) = 4 - x^2,\ g(x) = \sqrt{x + 1}$
   (a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 3} g(x)$ (c) $\lim_{x \to 1} g(f(x))$

26. $f(x) = 2x^2 - 3x + 1,\ g(x) = \sqrt[3]{x + 6}$
   (a) $\lim_{x \to 4} f(x)$ (b) $\lim_{x \to 21} g(x)$ (c) $\lim_{x \to 4} g(f(x))$

In Exercises 27–36, find the limit of the trigonometric function.

27. $\lim_{x \to \pi/2} \sin x$
28. $\lim_{x \to \pi} \tan x$
29. $\lim_{x \to 1} \cos \dfrac{\pi x}{3}$
30. $\lim_{x \to 2} \sin \dfrac{\pi x}{2}$
31. $\lim_{x \to 0} \sec 2x$
32. $\lim_{x \to \pi} \cos 3x$
33. $\lim_{x \to 5\pi/6} \sin x$
34. $\lim_{x \to 5\pi/3} \cos x$
35. $\lim_{x \to 3} \tan\left(\dfrac{\pi x}{4}\right)$
36. $\lim_{x \to 7} \sec\left(\dfrac{\pi x}{6}\right)$

In Exercises 37–40, use the information to evaluate the limits.

37. $\lim_{x \to c} f(x) = 3$
   $\lim_{x \to c} g(x) = 2$
   (a) $\lim_{x \to c} [5g(x)]$
   (b) $\lim_{x \to c} [f(x) + g(x)]$
   (c) $\lim_{x \to c} [f(x)g(x)]$
   (d) $\lim_{x \to c} \dfrac{f(x)}{g(x)}$

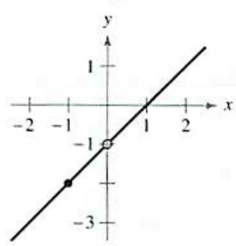
38. $\lim_{x \to c} f(x) = \frac{3}{2}$
   $\lim_{x \to c} g(x) = \frac{1}{2}$
   (a) $\lim_{x \to c} [4f(x)]$
   (b) $\lim_{x \to c} [f(x) + g(x)]$
   (c) $\lim_{x \to c} [f(x)g(x)]$
   (d) $\lim_{x \to c} \dfrac{f(x)}{g(x)}$

39. $\lim_{x \to c} f(x) = 4$
   (a) $\lim_{x \to c} [f(x)]^3$
   (b) $\lim_{x \to c} \sqrt{f(x)}$
   (c) $\lim_{x \to c} [3f(x)]$
   (d) $\lim_{x \to c} [f(x)]^{3/2}$

40. $\lim_{x \to c} f(x) = 27$
   (a) $\lim_{x \to c} \sqrt[3]{f(x)}$
   (b) $\lim_{x \to c} \dfrac{f(x)}{18}$
   (c) $\lim_{x \to c} [f(x)]^2$
   (d) $\lim_{x \to c} [f(x)]^{2/3}$

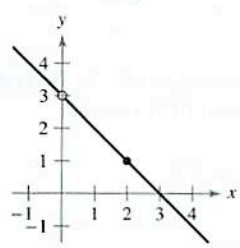
In Exercises 41–44, use the graph to determine the limit visually (if it exists). Write a simpler function that agrees with the given function at all but one point.

41. $g(x) = \dfrac{x^2 - x}{x}$



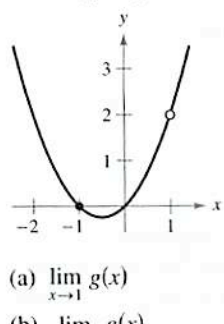
   (a) $\lim_{x \to 0} g(x)$
   (b) $\lim_{x \to -1} g(x)$

42. $h(x) = \dfrac{-x^2 + 3x}{x}$



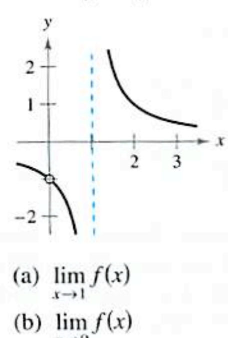
   (a) $\lim_{x \to 2} h(x)$
   (b) $\lim_{x \to 0} h(x)$

43. $g(x) = \dfrac{x^3 - x}{x - 1}$



   (a) $\lim_{x \to 1} g(x)$
   (b) $\lim_{x \to -1} g(x)$

44. $f(x) = \dfrac{x}{x^2 - x}$



   (a) $\lim_{x \to 1} f(x)$
   (b) $\lim_{x \to 0} f(x)$

In Exercises 45–48, find the limit of the function (if it exists). Write a simpler function that agrees with the given function at all but one point. Use a graphing utility to confirm your result.

45. $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$

46. $\lim_{x \to -1} \frac{2x^2 - x - 3}{x + 1}$

47. $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

48. $\lim_{x \to -1} \frac{x^3 + 1}{x + 1}$

In Exercises 49–64, find the limit (if it exists).

49. $\lim_{x \to 0} \frac{x}{x^2 - x}$

50. $\lim_{x \to 0} \frac{3x}{x^2 + 2x}$

51. $\lim_{x \to 4} \frac{x - 4}{x^2 - 16}$

52. $\lim_{x \to 3} \frac{3 - x}{x^2 - 9}$

53. $\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 - 9}$

54. $\lim_{x \to 4} \frac{x^2 - 5x + 4}{x^2 - 2x - 8}$

55. $\lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4}$

56. $\lim_{x \to 3} \frac{\sqrt{x + 1} - 2}{x - 3}$

57. $\lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x}$

58. $\lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x}$

59. $\lim_{x \to 0} \frac{[1/(3 + x)] - (1/3)}{x}$

60. $\lim_{x \to 0} \frac{[1/(x + 4)] - (1/4)}{x}$

61. $\lim_{\Delta x \to 0} \frac{2(x + \Delta x) - 2x}{\Delta x}$

62. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$

63. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x}$

64. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$

In Exercises 65–76, determine the limit of the trigonometric function (if it exists).

65. $\lim_{x \to 0} \frac{\sin x}{5x}$

66. $\lim_{x \to 0} \frac{3(1 - \cos x)}{x}$

67. $\lim_{x \to 0} \frac{\sin x(1 - \cos x)}{x^2}$

68. $\lim_{\theta \to 0} \frac{\cos \theta \tan \theta}{\theta}$

69. $\lim_{x \to 0} \frac{\sin^2 x}{x}$

70. $\lim_{x \to 0} \frac{\tan^2 x}{x}$

71. $\lim_{h \to 0} \frac{(1 - \cos h)^2}{h}$

72. $\lim_{\phi \to \pi} \phi \sec \phi$

73. $\lim_{x \to \pi/2} \frac{\cos x}{\cot x}$

74. $\lim_{x \to \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

75. $\lim_{t \to 0} \frac{\sin 3t}{2t}$

76. $\lim_{x \to 0} \frac{\sin 2x}{\sin 3x}$ $\left[\textit{Hint}: \text{Find } \lim_{x \to 0} \left(\frac{2 \sin 2x}{2x}\right)\left(\frac{3x}{3 \sin 3x}\right).\right]$

Graphical, Numerical, and Analytic Analysis **In Exercises 77–84, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

77. $\lim_{x \to 0} \frac{\sqrt{x + 2} - \sqrt{2}}{x}$

78. $\lim_{x \to 16} \frac{4 - \sqrt{x}}{x - 16}$

79. $\lim_{x \to 0} \frac{[1/(2 + x)] - (1/2)}{x}$

80. $\lim_{x \to 2} \frac{x^5 - 32}{x - 2}$

81. $\lim_{t \to 0} \frac{\sin 3t}{t}$

82. $\lim_{x \to 0} \frac{\cos x - 1}{2x^2}$

83. $\lim_{x \to 0} \frac{\sin x^2}{x}$

84. $\lim_{x \to 0} \frac{\sin x}{\sqrt[3]{x}}$

In Exercises 85–88, find $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$.

85. $f(x) = 3x - 2$

86. $f(x) = \sqrt{x}$

87. $f(x) = \frac{1}{x + 3}$

88. $f(x) = x^2 - 4x$

In Exercises 89 and 90, use the Squeeze Theorem to find $\lim_{x \to c} f(x)$.

89. $c = 0$

$4 - x^2 \leq f(x) \leq 4 + x^2$

90. $c = a$

$b - |x - a| \leq f(x) \leq b + |x - a|$

In Exercises 91–96, use a graphing utility to graph the given function and the equations $y = |x|$ **and** $y = -|x|$ **in the same viewing window. Using the graphs to observe the Squeeze Theorem visually, find** $\lim_{x \to 0} f(x)$.

91. $f(x) = x \cos x$

92. $f(x) = |x \sin x|$

93. $f(x) = |x| \sin x$

94. $f(x) = |x| \cos x$

95. $f(x) = x \sin \frac{1}{x}$

96. $h(x) = x \cos \frac{1}{x}$

WRITING ABOUT CONCEPTS

97. In the context of finding limits, discuss what is meant by two functions that agree at all but one point.

98. Give an example of two functions that agree at all but one point.

99. What is meant by an indeterminate form?

100. In your own words, explain the Squeeze Theorem.

101. *Writing* Use a graphing utility to graph

$$f(x) = x, \quad g(x) = \sin x, \quad \text{and} \quad h(x) = \frac{\sin x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x \to 0} h(x) = 1.$$

102. ***Writing*** Use a graphing utility to graph

$$f(x) = x,\ g(x) = \sin^2 x,\ \text{and } h(x) = \frac{\sin^2 x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why $\lim_{x \to 0} h(x) = 0$.

Free-Falling Object **In Exercises 103 and 104, use the position function $s(t) = -16t^2 + 500$, which gives the height (in feet) of an object that has fallen for t seconds from a height of 500 feet. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

103. If a construction worker drops a wrench from a height of 500 feet, how fast will the wrench be falling after 2 seconds?

104. If a construction worker drops a wrench from a height of 500 feet, when will the wrench hit the ground? At what velocity will the wrench impact the ground?

Free-Falling Object **In Exercises 105 and 106, use the position function $s(t) = -4.9t^2 + 200$, which gives the height (in meters) of an object that has fallen from a height of 200 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

105. Find the velocity of the object when $t = 3$.

106. At what velocity will the object impact the ground?

107. Find two functions f and g such that $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist, but $\lim_{x \to 0} [f(x) + g(x)]$ does exist.

108. Prove that if $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} [f(x) + g(x)]$ does not exist, then $\lim_{x \to c} g(x)$ does not exist.

109. Prove Property 1 of Theorem 1.1.

110. Prove Property 3 of Theorem 1.1. (You may use Property 3 of Theorem 1.2.)

111. Prove Property 1 of Theorem 1.2.

112. Prove that if $\lim_{x \to c} f(x) = 0$, then $\lim_{x \to c} |f(x)| = 0$.

113. Prove that if $\lim_{x \to c} f(x) = 0$ and $|g(x)| \le M$ for a fixed number M and all $x \neq c$, then $\lim_{x \to c} f(x)g(x) = 0$.

114. (a) Prove that if $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$.

(*Note:* This is the converse of Exercise 112.)

(b) Prove that if $\lim_{x \to c} f(x) = L$, then $\lim_{x \to c} |f(x)| = |L|$.

[*Hint:* Use the inequality $\|f(x)| - |L\| \le |f(x) - L|$.]

115. ***Think About It*** Find a function f to show that the converse of Exercise 114(b) is not true. [*Hint:* Find a function f such that $\lim_{x \to c} |f(x)| = |L|$ but $\lim_{x \to c} f(x)$ does not exist.]

CAPSTONE

116. Let $f(x) = \begin{cases} 3, & x \neq 2 \\ 5, & x = 2 \end{cases}$. Find $\lim_{x \to 2} f(x)$.

True or False? **In Exercises 117–122, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

117. $\lim_{x \to 0} \frac{|x|}{x} = 1$

118. $\lim_{x \to \pi} \frac{\sin x}{x} = 1$

119. If $f(x) = g(x)$ for all real numbers other than $x = 0$, and $\lim_{x \to 0} f(x) = L$, then $\lim_{x \to 0} g(x) = L$.

120. If $\lim_{x \to c} f(x) = L$, then $f(c) = L$.

121. $\lim_{x \to 2} f(x) = 3$, where $f(x) = \begin{cases} 3, & x \le 2 \\ 0, & x > 2 \end{cases}$

122. If $f(x) < g(x)$ for all $x \neq a$, then

$\lim_{x \to a} f(x) < \lim_{x \to a} g(x)$.

123. Prove the second part of Theorem 1.9.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

124. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

and

$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational.} \end{cases}$

Find (if possible) $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$.

125. ***Graphical Reasoning*** Consider $f(x) = \dfrac{\sec x - 1}{x^2}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph f. Is the domain of f obvious from the graph? If not, explain.

(c) Use the graph of f to approximate $\lim_{x \to 0} f(x)$.

(d) Confirm your answer to part (c) analytically.

126. ***Approximation***

(a) Find $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$.

(b) Use your answer to part (a) to derive the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ for x near 0.

(c) Use your answer to part (b) to approximate $\cos(0.1)$.

(d) Use a calculator to approximate $\cos(0.1)$ to four decimal places. Compare the result with part (c).

127. ***Think About It*** When using a graphing utility to generate a table to approximate $\lim_{x \to 0} [(\sin x)/x]$, a student concluded that the limit was 0.01745 rather than 1. Determine the probable cause of the error.