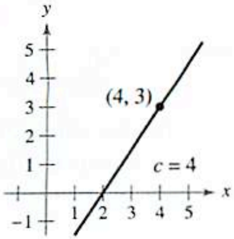
1.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

(a) $\lim_{x \to c^+} f(x)$ (b) $\lim_{x \to c^-} f(x)$ (c) $\lim_{x \to c} f(x)$

1.



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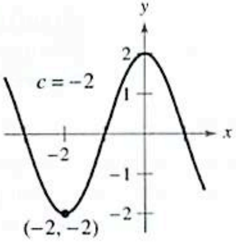

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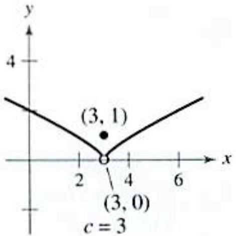

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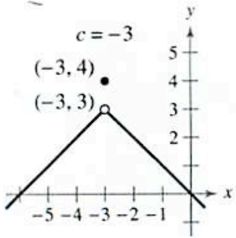

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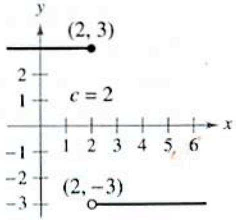

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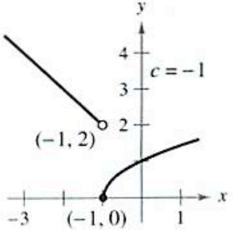


In Exercises 7–26, find the limit (if it exists). If it does not exist, explain why.

7. $\lim\limits_{x\to 8^+} \dfrac{1}{x+8}$

8. $\lim\limits_{x\to 5^-} -\dfrac{3}{x+5}$

9. $\lim\limits_{x\to 5^+} \dfrac{x-5}{x^2-25}$

10. $\lim\limits_{x\to 2^+} \dfrac{2-x}{x^2-4}$

11. $\lim\limits_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$

12. $\lim\limits_{x\to 9^-} \dfrac{\sqrt{x}-3}{x-9}$

13. $\lim\limits_{x\to 0^-} \dfrac{|x|}{x}$

14. $\lim\limits_{x\to 10^+} \dfrac{|x-10|}{x-10}$

15. $\lim\limits_{\Delta x\to 0^-} \dfrac{\dfrac{1}{x+\Delta x} - \dfrac{1}{x}}{\Delta x}$

16. $\lim\limits_{\Delta x\to 0^+} \dfrac{(x+\Delta x)^2 + x + \Delta x - (x^2+x)}{\Delta x}$

17. $\lim\limits_{x\to 3^-} f(x)$, where $f(x) = \begin{cases} \dfrac{x+2}{2}, & x \le 3 \\ \dfrac{12-2x}{3}, & x > 3 \end{cases}$

18. $\lim\limits_{x\to 2} f(x)$, where $f(x) = \begin{cases} x^2-4x+6, & x < 2 \\ -x^2+4x-2, & x \ge 2 \end{cases}$

19. $\lim\limits_{x\to 1} f(x)$, where $f(x) = \begin{cases} x^3+1, & x < 1 \\ x+1, & x \ge 1 \end{cases}$

20. $\lim\limits_{x\to 1^+} f(x)$, where $f(x) = \begin{cases} x, & x \le 1 \\ 1-x, & x > 1 \end{cases}$

21. $\lim\limits_{x\to \pi} \cot x$

22. $\lim\limits_{x\to \pi/2} \sec x$

23. $\lim\limits_{x\to 4^-} (5[\![x]\!] - 7)$

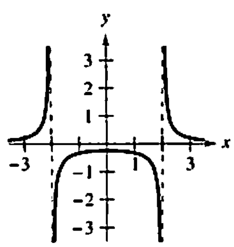
24. $\lim\limits_{x\to 2^+} (2x - [\![x]\!])$

25. $\lim\limits_{x\to 3} (2 - [\![-x]\!])$

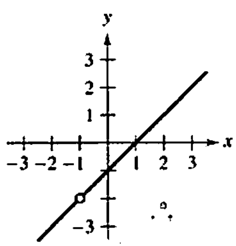
26. $\lim\limits_{x\to 1} \left(1 - \left[\!\!\left[-\dfrac{x}{2}\right]\!\!\right]\right)$

In Exercises 27–30, discuss the continuity of each function.

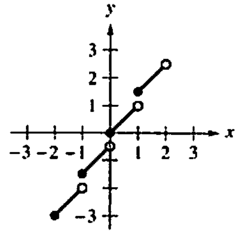
27. $f(x) = \dfrac{1}{x^2-4}$



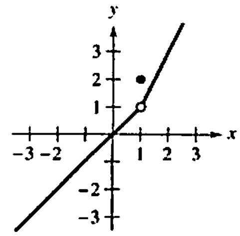
28. $f(x) = \dfrac{x^2-1}{x+1}$



29. $f(x) = \frac{1}{2}[\![x]\!] + x$



30. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x-1, & x > 1 \end{cases}$



In Exercises 31–34, discuss the continuity of the function on the closed interval.

	Function	*Interval*
31.	$g(x) = \sqrt{49-x^2}$	$[-7, 7]$
32.	$f(t) = 3 - \sqrt{9-t^2}$	$[-3, 3]$
33.	$f(x) = \begin{cases} 3-x, & x \le 0 \\ 3+\frac{1}{2}x, & x > 0 \end{cases}$	$[-1, 4]$
34.	$g(x) = \dfrac{1}{x^2-4}$	$[-1, 2]$

In Exercises 35–60, find the x-values (if any) at which f is not continuous. Which of the discontinuities are removable?

35. $f(x) = \dfrac{6}{x}$

36. $f(x) = \dfrac{3}{x-2}$

37. $f(x) = x^2 - 9$

38. $f(x) = x^2 - 2x + 1$

39. $f(x) = \dfrac{1}{4-x^2}$

40. $f(x) = \dfrac{1}{x^2+1}$

41. $f(x) = 3x - \cos x$

42. $f(x) = \cos\dfrac{\pi x}{2}$

43. $f(x) = \dfrac{x}{x^2-x}$

44. $f(x) = \dfrac{x}{x^2-1}$

45. $f(x) = \dfrac{x}{x^2+1}$

46. $f(x) = \dfrac{x-6}{x^2-36}$

47. $f(x) = \dfrac{x+2}{x^2-3x-10}$

48. $f(x) = \dfrac{x-1}{x^2+x-2}$

49. $f(x) = \dfrac{|x+7|}{x+7}$

50. $f(x) = \dfrac{|x-8|}{x-8}$

51. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

52. $f(x) = \begin{cases} -2x+3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

53. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

54. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

55. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

56. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

57. $f(x) = \csc 2x$ **58.** $f(x) = \tan \dfrac{\pi x}{2}$

59. $f(x) = [\![x - 8]\!]$ **60.** $f(x) = 5 - [\![x]\!]$

In Exercises 61 and 62, use a graphing utility to graph the function. From the graph, estimate

$$\lim_{x\to 0^+} f(x) \quad \text{and} \quad \lim_{x\to 0^-} f(x).$$

Is the function continuous on the entire real line? Explain.

61. $f(x) = \dfrac{|x^2 - 4|x}{x + 2}$ **62.** $f(x) = \dfrac{|x^2 + 4x|(x + 2)}{x + 4}$

In Exercises 63–68, find the constant a, or the constants a and b, such that the function is continuous on the entire real line.

63. $f(x) = \begin{cases} 3x^2, & x \ge 1 \\ ax - 4, & x < 1 \end{cases}$

64. $f(x) = \begin{cases} 3x^3, & x \le 1 \\ ax + 5, & x > 1 \end{cases}$

65. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

66. $g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

67. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

68. $g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

In Exercises 69–72, discuss the continuity of the composite function $h(x) = f(g(x))$.

69. $f(x) = x^2$
$g(x) = x - 1$

70. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = x - 1$

71. $f(x) = \dfrac{1}{x - 6}$
$g(x) = x^2 + 5$

72. $f(x) = \sin x$
$g(x) = x^2$

In Exercises 73–76, use a graphing utility to graph the function. Use the graph to determine any x-values at which the function is not continuous.

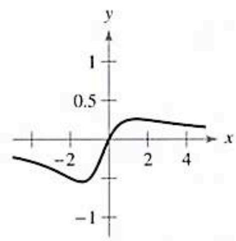
73. $f(x) = [\![x]\!] - x$ **74.** $h(x) = \dfrac{1}{x^2 - x - 2}$

75. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

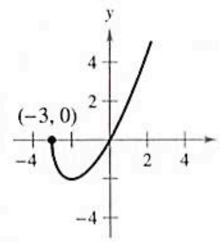
76. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

In Exercises 77–80, describe the interval(s) on which the function is continuous.

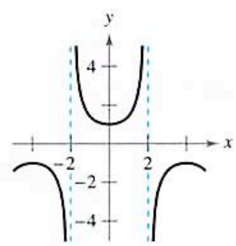
77. $f(x) = \dfrac{x}{x^2 + x + 2}$



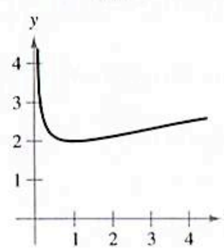
78. $f(x) = x\sqrt{x + 3}$



79. $f(x) = \sec \dfrac{\pi x}{4}$



80. $f(x) = \dfrac{x + 1}{\sqrt{x}}$



Writing **In Exercises 81 and 82, use a graphing utility to graph the function on the interval $[-4, 4]$. Does the graph of the function appear to be continuous on this interval? Is the function continuous on $[-4, 4]$? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

81. $f(x) = \dfrac{\sin x}{x}$ **82.** $f(x) = \dfrac{x^3 - 8}{x - 2}$

Writing **In Exercises 83–86, explain why the function has a zero in the given interval.**

Function	*Interval*
83. $f(x) = \frac{1}{12}x^4 - x^3 + 4$	$[1, 2]$
84. $f(x) = x^3 + 5x - 3$	$[0, 1]$
85. $f(x) = x^2 - 2 - \cos x$	$[0, \pi]$
86. $f(x) = -\dfrac{5}{x} + \tan\left(\dfrac{\pi x}{10}\right)$	$[1, 4]$

In Exercises 87–90, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval [0, 1]. Repeatedly "zoom in" on the graph of the function to approximate the zero accurate to two decimal places. Use the *zero* or *root* feature of the graphing utility to approximate the zero accurate to four decimal places.

87. $f(x) = x^3 + x - 1$

88. $f(x) = x^3 + 5x - 3$

89. $g(t) = 2\cos t - 3t$

90. $h(\theta) = 1 + \theta - 3\tan\theta$

In Exercises 91–94, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of c guaranteed by the theorem.

91. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$

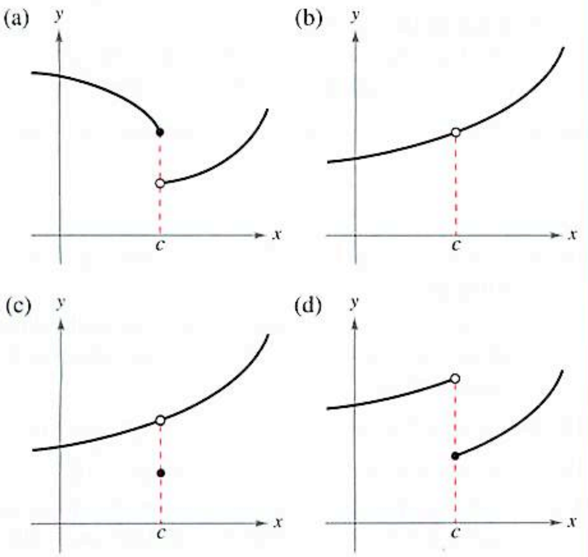
92. $f(x) = x^2 - 6x + 8, \quad [0, 3], \quad f(c) = 0$

93. $f(x) = x^3 - x^2 + x - 2, \quad [0, 3], \quad f(c) = 4$

94. $f(x) = \dfrac{x^2 + x}{x - 1}, \quad \left[\dfrac{5}{2}, 4\right], \quad f(c) = 6$

WRITING ABOUT CONCEPTS

95. State how continuity is destroyed at $x = c$ for each of the following graphs.



96. Sketch the graph of any function f such that

$$\lim_{x\to 3^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 3^-} f(x) = 0.$$

Is the function continuous at $x = 3$? Explain.

97. If the functions f and g are continuous for all real x, is $f + g$ always continuous for all real x? Is f/g always continuous for all real x? If either is not continuous, give an example to verify your conclusion.

CAPSTONE

98. Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following descriptions.

(a) A function with a nonremovable discontinuity at $x = 4$

(b) A function with a removable discontinuity at $x = -4$

(c) A function that has both of the characteristics described in parts (a) and (b)

True or False? **In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

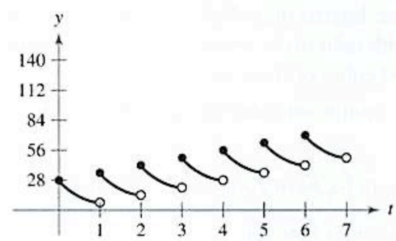
99. If $\lim_{x\to c} f(x) = L$ and $f(c) = L$, then f is continuous at c.

100. If $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$, then either f or g is not continuous at c.

101. A rational function can have infinitely many x-values at which it is not continuous.

102. The function $f(x) = |x - 1|/(x - 1)$ is continuous on $(-\infty, \infty)$.

103. ***Swimming Pool*** Every day you dissolve 28 ounces of chlorine in a swimming pool. The graph shows the amount of chlorine $f(t)$ in the pool after t days.



Estimate and interpret $\lim_{t\to 4^-} f(t)$ and $\lim_{t\to 4^+} f(t)$.

104. ***Think About It*** Describe how the functions

$f(x) = 3 + [\![x]\!]$

and

$g(x) = 3 - [\![-x]\!]$

differ.

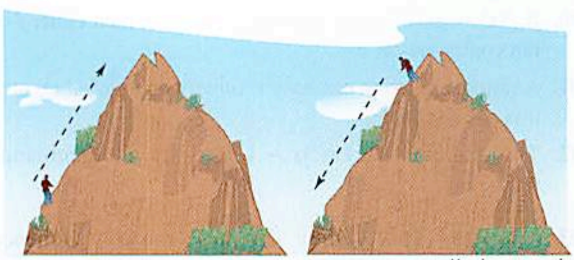
105. ***Telephone Charges*** A long distance phone service charges \$0.40 for the first 10 minutes and \$0.05 for each additional minute or fraction thereof. Use the greatest integer function to write the cost C of a call in terms of time t (in minutes). Sketch the graph of this function and discuss its continuity.

106. ***Inventory Management*** The number of units in inventory in a small company is given by

$$N(t) = 25\left(2\left[\!\left[\frac{t+2}{2}\right]\!\right] - t\right)$$

where t is the time in months. Sketch the graph of this function and discuss its continuity. How often must this company replenish its inventory?

107. ***Déjà Vu*** At 8:00 A.M. on Saturday a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at 8:00 A.M. he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct. [*Hint:* Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]



Not drawn to scale

Saturday 8:00 A.M. Sunday 8:00 A.M.

108. ***Volume*** Use the Intermediate Value Theorem to show that for all spheres with radii in the interval $[5, 8]$, there is one with a volume of 1500 cubic centimeters.

109. Prove that if f is continuous and has no zeros on $[a, b]$, then either

$f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

110. Show that the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any real number.

111. Show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ kx, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$. (Assume that k is any nonzero real number.)

112. The **signum function** is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Sketch a graph of $\operatorname{sgn}(x)$ and find the following (if possible).

(a) $\lim\limits_{x\to 0^-} \operatorname{sgn}(x)$ (b) $\lim\limits_{x\to 0^+} \operatorname{sgn}(x)$ (c) $\lim\limits_{x\to 0} \operatorname{sgn}(x)$

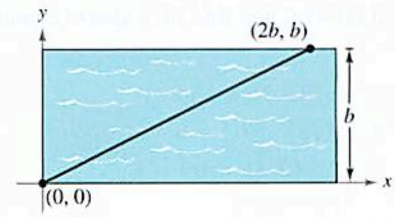
113. ***Modeling Data*** The table lists the speeds S (in feet per second) of a falling object at various times t (in seconds).

t	0	5	10	15	20	25	30
S	0	48.2	53.5	55.2	55.9	56.2	56.3

(a) Create a line graph of the data.

(b) Does there appear to be a limiting speed of the object? If there is a limiting speed, identify a possible cause.

114. ***Creating Models*** A swimmer crosses a pool of width b by swimming in a straight line from $(0, 0)$ to $(2b, b)$. (See figure.)



(a) Let f be a function defined as the y-coordinate of the point on the long side of the pool that is nearest the swimmer at any given time during the swimmer's crossing of the pool. Determine the function f and sketch its graph. Is f continuous? Explain.

(b) Let g be the minimum distance between the swimmer and the long sides of the pool. Determine the function g and sketch its graph. Is g continuous? Explain.

115. Find all values of c such that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$$

116. Prove that for any real number y there exists x in $(-\pi/2, \pi/2)$ such that $\tan x = y$.

117. Let $f(x) = \left(\sqrt{x + c^2} - c\right)/x$, $c > 0$. What is the domain of f? How can you define f at $x = 0$ in order for f to be continuous there?

118. Prove that if $\lim\limits_{\Delta x\to 0} f(c + \Delta x) = f(c)$, then f is continuous at c.

119. Discuss the continuity of the function $h(x) = x[\![x]\!]$.

120. (a) Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interval $[a, b]$. If $f_1(a) < f_2(a)$ and $f_1(b) > f_2(b)$, prove that there exists c between a and b such that $f_1(c) = f_2(c)$.

(b) Show that there exists c in $\left[0, \frac{\pi}{2}\right]$ such that $\cos x = x$. Use a graphing utility to approximate c to three decimal places.

PUTNAM EXAM CHALLENGE

121. Prove or disprove: if x and y are real numbers with $y \ge 0$ and $y(y + 1) \le (x + 1)^2$, then $y(y - 1) \le x^2$.

122. Determine all polynomials $P(x)$ such that

$P(x^2 + 1) = (P(x))^2 + 1$ and $P(0) = 0$.