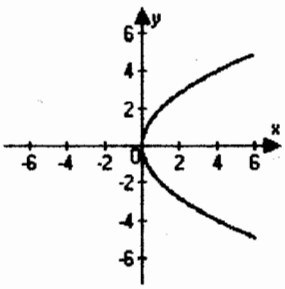
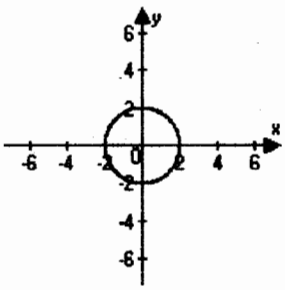
M251 practice exam 6 9.1-9.5
P. Staley

1. Match the equation with its graph. $y^2 = 4x$

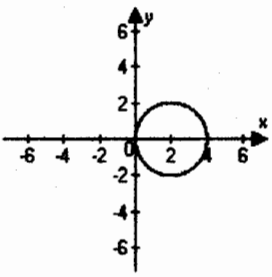
A)



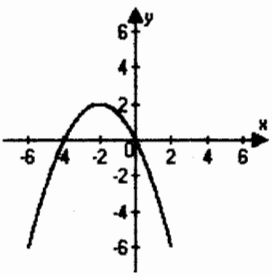
B)



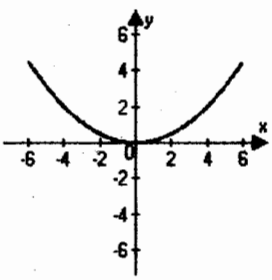
C)



D)

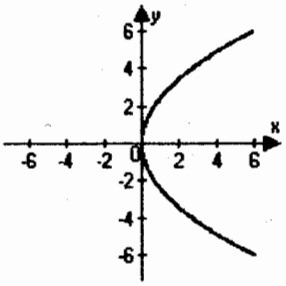


E)



2. Match the equation with its graph. $\frac{x^2}{9}+\frac{y^2}{9}=1$

A)



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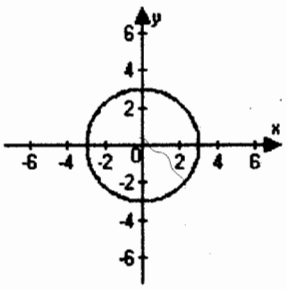

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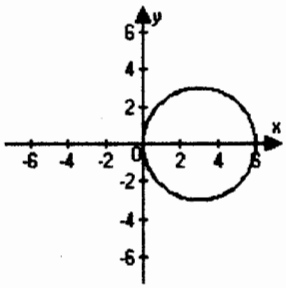
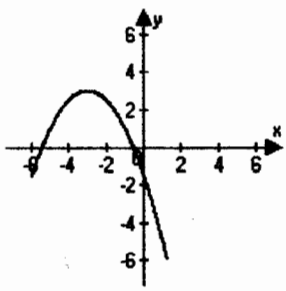


D)



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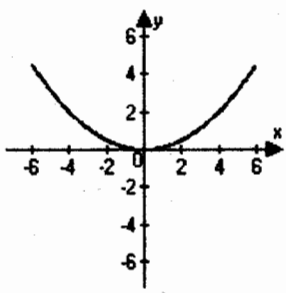
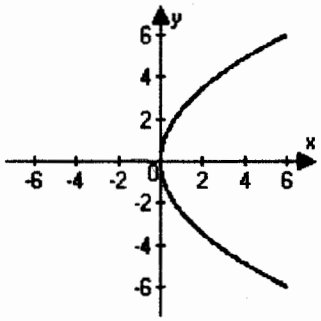


3. Match the equation with its graph. $\frac{(x-2)^2}{1}-\frac{y^2}{4}=1$

A)



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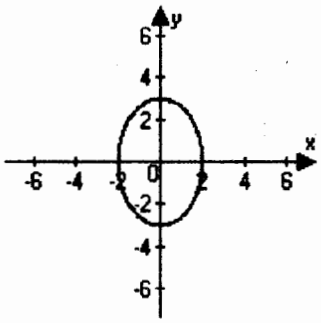
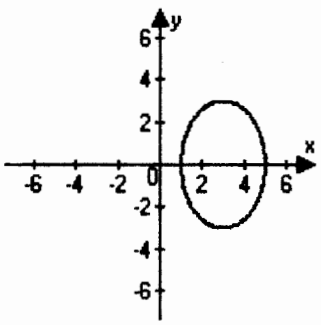
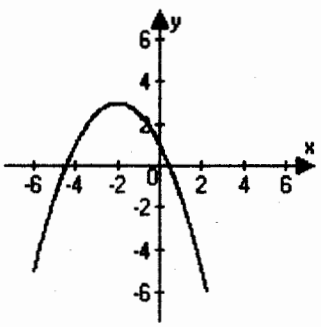


C)



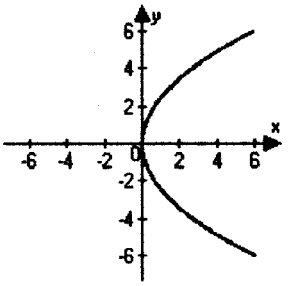
D)



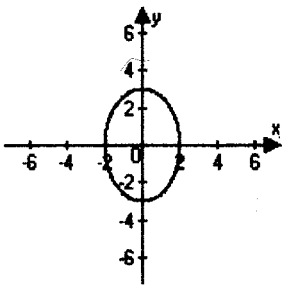
E) None of the above.

4. Match the equation with its graph. $(x+2)^2 = -2(y-3)$

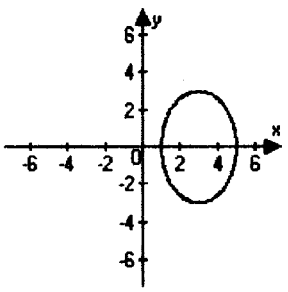
A)



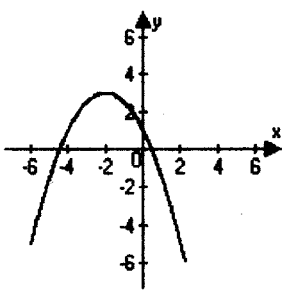
B)



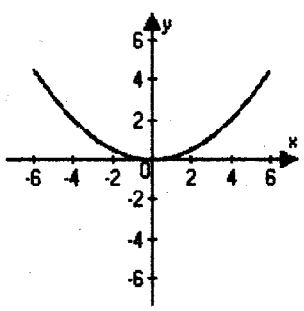
C)



D)



E)



5. Find the vertex, focus, and directrix of the parabola.

$y^2+8y+4x+12=0$

$y^2+8y+16-16+4x+12=0$

$(y+4)^2=-4(x+1)$

vertex: $(-1,-4)$ focus: $(-2,-4)$ directrix: $x=0$

6. Find an equation of the parabola with vertex (0,8) and directrix $y=-5$.

vertex $=\left(0, \frac{8-5}{2}\right)=\left(0, \frac{3}{2}\right)$ $p=8-\frac{3}{2}=\frac{13}{2}$

$4p(y-k)=(x-h)^2$ $\quad 4\left(\frac{13}{2}\right)\left(y-\frac{3}{2}\right)=x^2$

$26y-39=x^2$

7. Find the center, foci, vertices, and eccentricity of the ellipse.

$\frac{(x-1)^2}{25}+\frac{(y+5)^2}{9}=1$ $\qquad \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

$c^2+b^2=a^2$

center: (h,k)
vertices $(h\pm a,k)$
foci $(h\pm c,k)$
eccentricity: $\frac{c}{a}$

center: $(1,-5)$ vertices: $(-4,-5)$ $(6,-5)$ foci: $(-3,-5)$ $(5,-5)$ eccentricity: $\frac{4}{5}$

8. Find an equation of the ellipse with vertices (0,6), (14,6) and eccentricity $\varepsilon=\frac{1}{7}$.

$(h,k)=(7,6)$

$a=7$ $\qquad \frac{c}{7}=\frac{1}{7} \Rightarrow c=1$

$c^2+b^2=a^2 \Rightarrow b=\sqrt{49-1}=\sqrt{48}$

$\frac{(x-7)}{49}+\frac{(y-6)}{48}=1$

9. Find the center, foci, and vertices of the hyperbola.

$\frac{(x-1)^2}{9} - \frac{(y+2)^2}{4} = 1$

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$a^2+b^2=c^2$

$\sqrt{2^2+3^2}=c$

$c=\sqrt{13}$

center (h,k)

vertices $(h\pm a, k)$

foci $(h\pm c, k)$

center: $(1,-2)$ vertices: $(-2,-2)$ $(4,-2)$ foci: $(1\pm\sqrt{13}, -2)$

10. Find an equation of the hyperbola with vertices (-2,0), (2,0) and asymptotes $y=\pm 8x$.

center: $(0,0)$

$\frac{x^2}{4} - \frac{y^2}{256} = 1$

$a=2$ $y=\pm\frac{b}{a}x$

$8=\frac{b}{2} \Rightarrow b=16$

11. Find an equation of the hyperbola with vertices (0,–10), (0,10) and asymptotes $y=\pm\frac{1}{10}x$.

center: $(0,0)$ $a=10$

$y=\pm\frac{b}{a}x$ $\frac{b}{10}=\frac{1}{10} \Rightarrow b=1$ $\frac{y^2}{100} - x^2 = 1$

12. Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

$x^2+7y+9x-6=0$

A) Parabola
B) Circle
C) Ellipse
D) Hyperbola

13. Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

$6x^2+5y^2+7x+5y-3=0$

A) Parabola
B) Circle
C) Ellipse
D) Hyperbola

14. Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

$4x^2+4y^2+4x+9y-7=0$

A) Parabola
B) Circle
C) Ellipse
D) Hyperbola

15. Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

$9x^2-8y^2+6x+3y-1=0$

A) Parabola
B) Circle
C) Ellipse
D) Hyperbola

16. Write the corresponding rectangular equation by eliminating the parameter.

$x=2t-1$
$y=3t+1$

$\frac{x+1}{2}=t$

$\frac{y-1}{3}=t$

$\frac{x+1}{2}=\frac{y-1}{3}$

$3x+3=2y-2$

$2y=3x+5$

17. Write the corresponding rectangular equation by eliminating the parameter.

$x=\sqrt{t}$
$y=t-4$

$x^2=t \quad x\geq 0$

$y=x^2-4 \; ; \; x\geq 0$

18. Write the corresponding rectangular equation by eliminating the parameter.

$x=e^t$
$y=e^{2t}+1$

$y=(e^t)^2+1$

$y=x^2+1 \quad x>0$

19. Write the corresponding rectangular equation by eliminating the parameter.

$x = 4\cos\theta$
$y = 8\sin\theta$

$\frac{x}{4} = \cos\theta$

$\left(\frac{y}{8}\right) = \sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$$\frac{x^2}{16} + \frac{y^2}{64} = 1$$

20. Write the corresponding rectangular equation by eliminating the parameter.

$x = 16 + 8\cos\theta$
$y = -4 + 4\sin\theta$

$\frac{x-16}{8} = \cos\theta$

$\frac{y+4}{4} = \sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$$\frac{(x-16)^2}{64} + \frac{(y+4)^2}{16} = 1$$

21. Write the corresponding rectangular equation by eliminating the parameter.

$x = t^3$
$y = 3\ln t$

$\Rightarrow y = \ln(t^3) \Rightarrow$ $y = \ln x$

22. Find $\frac{dy}{dx}$.

$x = t^2$
$y = 8 - 10t$

$\frac{y-8}{-10} = t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-10}{2t} = \frac{-5}{t}$$

$$= \begin{cases} \frac{5}{\sqrt{x}} & \text{for } y \geq 8 \text{ i.e. } t \leq 0 \\ -\frac{5}{\sqrt{x}} & \text{for } y < 8 \text{ i.e. } t > 0 \end{cases}$$

23. Find $\frac{dy}{dx}$.

$x = \sqrt{t}$
$y = 6 - t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1}{\frac{1}{2\sqrt{t}}} = -2\sqrt{t} = -2x$$

24. Find $\frac{dy}{dx}$.

$x = 2e^{\theta}$

$y = e^{\frac{\theta}{2}}$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{1}{2}e^{\frac{\theta}{2}}}{2e^{\theta}} = \frac{\frac{1}{2}\sqrt{\frac{x}{2}}}{x} = \frac{1}{2\sqrt{2x}}$$

25. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if possible, and find the slope and concavity (if possible) at the point corresponding to $t = 4$.

$x = t + 6$

$y = t^2 + 4t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+4}{1} = 2(x-6)+4 = 2x-8$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{2}{1}$$

$dy/dx = 2x-8$ or $2t+4$ $\quad d^2y/dx^2 = 2$ $\quad$ At $t = 4$: slope: 12 $\quad$ and convave up

26. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if possible, and find the slope and concavity (if possible) at the point corresponding to $\theta = \frac{\pi}{4}$.

$x = 3\cos\theta$
$y = 3\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta}{-3\sin\theta} = -\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta} = \frac{-\csc^2\theta}{-3\sin\theta} = \frac{-\csc^3\theta}{3}$$

$dy/dx = -\cot\theta$ $d^2y/dx^2 = \frac{-\csc^3\theta}{3}$ at $\theta = \frac{\pi}{4}$: slope: $-\cot\frac{\pi}{4} = -1$ and concave downward

27. Find the arc length of the curve on the given interval.

$x = t^2$, $y = 20t$, $0 \le t \le 10$

$$\int_0^{10} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{10} \sqrt{4t^2 + 20^2}\,dt = 2\int_0^{10} \sqrt{t^2+100}\,dt$$

Formula 26 Pg A28

$$\left[t\sqrt{t^2+100} + 100\ln\left|t+\sqrt{t^2+100}\right|\right]_0^{10} = 10\sqrt{200} + 100\ln(10+\sqrt{200}) - 100\ln 10$$
$$= 100\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$$

28. Find the arc length of the curve on the given interval.

$x = t^2 + 6$, $y = 4t^3 + 10$, $-1 \le t \le 0$

$$\int_{-1}^{0} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_{-1}^{0} \sqrt{(2t)^2 + (12t^2)^2}\,dt = 2\int_{-1}^{0} |t|\sqrt{36t^2+1}\,dt$$

note $t < 0$ on $[-1, 0]$

$u = 36t^2 + 1$
$du = 72t\,dt$

$$\frac{2}{72}\int_1^{37} \sqrt{u}\,du = \left[\frac{2}{3\cdot 36}u^{3/2}\right]_1^{37} = \frac{1}{54}\left(37\sqrt{37} - 1\right)$$

29. Find the arc length of the curve on the given interval.

$x = \sqrt{t}$, $y = 8t - 6$, $0 \le t \le 8$

$$\int_0^8 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^8 \sqrt{\left(\frac{1}{2\sqrt{t}}\right)^2 + 64}\,dt = \frac{1}{2}\int_0^8 \sqrt{\frac{1}{t} + 256}\,dt$$

improper!

reparameterize with
$s^2 = t$
$x = s$
$y = 8s^2 - 6$
$0 \le s \le \sqrt{8}$

$$\int_0^{\sqrt{8}} \sqrt{1 + 256s^2}\,ds$$

use formula 26 pg A29 to get 64.1564

30. Find the area of the surface generated by revolving the curve about the given axis.

$x = t,\ y = 5t,\ 0 \le t \le 10$

(i) x-axis: $S = 2\pi \int_0^{10} 5t\sqrt{1^2+5^2}\,dt = 10\sqrt{26}\pi\left[\frac{t^2}{2}\right]_0^{10} = 500\pi\sqrt{26}$

(ii) y-axis: $S = 2\pi \int_0^{10} t\sqrt{1^2+5^2}\,dt = 100\pi\sqrt{26}$

31. Find the area of the surface generated by revolving the curve about the given axis.

$x = 9\cos^3\theta,\ y = 9\sin^3\theta,\ 0 \le \theta \le \pi/2$

(i) x-axis:

$S = 2\pi \int_0^{\frac{\pi}{2}} 9\sin^3\theta\sqrt{(-27\cos^2\theta\sin\theta)^2 + (27\sin^2\theta\cos\theta)^2}\,d\theta$

$= 2\cdot 9\cdot 27\pi \int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\sqrt{\cos^2\theta+\sin^2\theta}\,d\theta$

$u = \sin\theta$

$2\cdot 9\cdot 27\pi \int_0^1 u^4\,du = \frac{2\cdot 9\cdot 27\pi}{5}$

$= \frac{486}{5}\pi$

$= 97.2\pi$

(ii) y-axis:

$2\pi \int_0^{\frac{\pi}{2}} 9\cos^3\theta\sqrt{\quad}\,d\theta = 2\cdot 9\cdot 27\pi \int_0^{\frac{\pi}{2}} \cos^4\theta\sin\theta\,d\theta$

$u = \cos\theta$

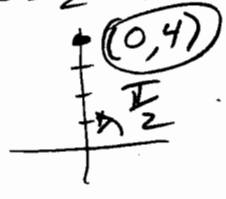
$2\cdot 9\cdot 27\pi \int_1^0 -u^4\,du = 97.2\pi$

32. For the given point in polar coordinates, find the corresponding rectangular coordinates for the point.

$\left(4, \frac{\pi}{2}\right)$

$\cos\frac{\pi}{2} = 0 \qquad \sin\frac{\pi}{2} = 1$

(0,4)

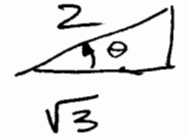


33. For the given point in rectangular coordinates, find two sets of polar coordinates for the point for $0 \le \theta \le 2\pi$.

$(8\sqrt{3}, 8)$

$x = r\cos\theta = 8\sqrt{3}$

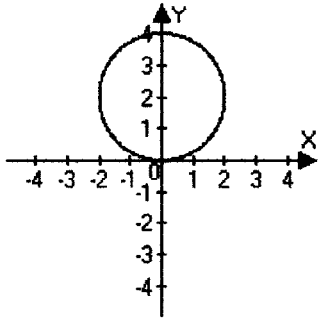
$y = r\sin\theta = 8\cdot 1$



zoom by 8

$r = 16 \qquad \theta = \frac{\pi}{6} \qquad \left(16, \frac{\pi}{6}\right)$ in polar

34. Specify the polar equation for the given graph.



$x^2 + (y-2)^2 = 4$

$x^2 + y^2 - 4y = 0$

$r^2 = 4r\sin\theta$

divide by r

$r = 4\sin\theta$

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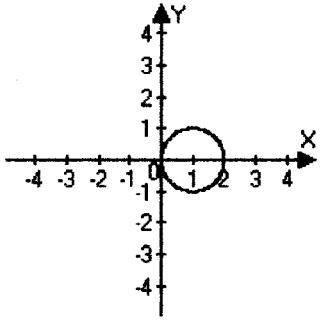
10:59 pm, 9/14/06

35. Specify the polar equation for the given graph.



$(x-1)^2 + y^2 = 1$

$x^2 - 2x + 1 + y^2 = 1$

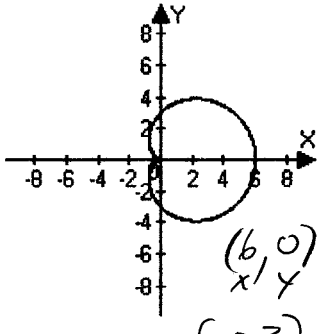
$x^2 + y^2 = 2x$

$r^2 = 2r\cos\theta$

divide by r

$r = 2\cos\theta$

36. Specify the polar equation for the given graph.



using information on page 690

this is a cardioid

$r = a \pm a\cos\theta$ or $r = a \pm a\sin\theta$

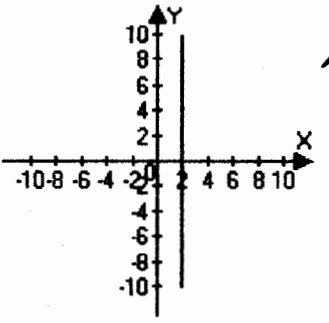
$(6,0) \Rightarrow r = 3 + 3\cos\theta$ or $r = 6 \pm 6\sin\theta$

x/y

NO WAY

$(0,3) \Rightarrow r = 3 + 3\cos\theta$

37. Specify the polar equation for the given graph.



$x = 2 \Rightarrow r\cos\theta = 2$

$\Rightarrow r = 2\sec\theta$

38. Convert the rectangular equation to polar form.

$x = 5$ $\quad r\cos\theta = 5 \Rightarrow r = 5\sec\theta$

39. Convert the rectangular equation to polar form.

$9x - y + 8 = 0$ $\quad 9r\cos\theta - r\sin\theta + 8 = 0$

$r(9\cos\theta - \sin\theta) = -8$

$r = \dfrac{8}{\sin\theta - 9\cos\theta}$

40. Convert the polar equation to rectangular form.

$r = 6$ $\quad \sqrt{x^2+y^2} = 6 \Rightarrow x^2 + y^2 = 36$

41. Convert the polar equation to rectangular form.

$r = 10\sin\theta$ $\quad r^2 = 10r\sin\theta$

$x^2 + y^2 = 10y$

$x^2 + y^2 - 10y = 0$

$x^2 + y^2 - 10y + 25 = 25$

$x^2 + (y-5)^2 = 5^2$

42. Find the points of intersection of the graphs of the equations.

$r = 1+\cos\theta$
$r = 3\cos\theta$

obvious intersections

$1+\cos\theta = 3\cos\theta$

$\frac{1}{2} = \cos\theta$

$\Rightarrow \pm\frac{\pi}{3} \Rightarrow \left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, -\frac{\pi}{3}\right)$

obscure intersection at $(0,0)$
ie $(0,\pi)$ and $\left(0,\frac{\pi}{2}\right)$
$= (0,0)$ in rectangular

43. Find the points of intersection of the graphs of the equations.

$r = \frac{\theta}{2.4} \Rightarrow \theta = (2.4)^2 \Rightarrow (2.4, (2.4)^2)$

$r = 2.4$

but 2nd graph is also $r = -2.4$
which gives the additional intersection $(-2.4, -(2.4)^2)$

44. Find the length of the curve over the given interval.

$r = 20\cos\theta, \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{f^2 + f'^2}\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(20\cos\theta)^2 + (-20\sin\theta)^2}\, d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 20\, d\theta = 20\pi$$

45. Find the length of the curve over the given interval.

$r = 5+5\sin\theta, \ 0 \le \theta \le 2\pi$

$$\int_0^{2\pi} \sqrt{f^2 + f'^2}\, d\theta = \int_0^{2\pi} \sqrt{(5+5\sin\theta)^2 + (5\cos\theta)^2}\, d\theta$$

$$5\int_0^{2\pi} \sqrt{2+2\sin\theta}\, d\theta = 10\int_0^{2\pi} \left|\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right| d\theta = 40$$

46. Find the length of the curve over the given interval.

$r = 9(1+\cos\theta), \ 0 \le \theta \le 2\pi$

$$\int_0^{2\pi} \sqrt{f^2 + f'^2}\, d\theta = 9\int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta}\, d\theta$$

$$9\int_0^{2\pi} \sqrt{2+2\cos\theta}\, d\theta = 18\int_0^{2\pi} \sqrt{\cos^2\frac{\theta}{2}} = 18\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta = 72$$

47. Find the area of the surface formed by revolving about the *polar axis* the following curve over the given interval.

revolve about polar axis $\Rightarrow$ radius $= y = f(\theta)\sin\theta$

$r = 8\cos\theta,\ 0 \le \theta \le \frac{\pi}{2}$

$$2\pi\int_0^{\frac{\pi}{2}} 8\cos\theta\sin\theta\sqrt{(8\cos\theta)^2 + (-8\sin\theta)^2}\,d\theta \quad \left(\begin{array}{c}\text{see thm 9.15}\\ \text{pg 699}\end{array}\right)$$

$$128\pi\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = 128\pi\int_0^1 u\,du = 128\pi\left[\frac{u^2}{2}\right]_0^1 = 64\pi$$

$u = \sin\theta$, $du = \cos\theta\,d\theta$

48. Find the area of the surface formed by revolving about the $\theta = \frac{\pi}{2}$ axis the following curve over the given interval.

$\Rightarrow$ radius $= x = f(\theta)\cos\theta$

$r = e^{5\theta},\ 0 \le \theta \le \frac{\pi}{2}$

$$2\pi\int_0^{\frac{\pi}{2}} e^{5\theta}\cos\theta\sqrt{(e^{5\theta})^2 + (5e^{5\theta})^2}\,d\theta$$

$$2\pi\int_0^{\frac{\pi}{2}} e^{10\theta}\cos\theta\sqrt{26}\,d\theta$$

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$$2\pi\sqrt{26}\left[\frac{e^{10\theta}}{1+100}\left(10\cos\theta + \sin\theta\right)\right]_0^{\frac{\pi}{2}} = 2\pi\sqrt{26}\left(\frac{e^{5\pi}}{101} - \frac{10}{101}\right)$$

$$= \frac{2\pi\sqrt{26}}{101}\left(e^{5\pi} - 10\right)$$