1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} \qquad (n \neq -1)$

2. $\int \frac{dx}{x} = \log_e |x| \qquad (x \neq 0)$

3. $\int \sin x \, dx = -\cos x$

4. $\int \cos x \, dx = \sin x$

5. $\int \tan x \, dx = -\log_e |\cos x|$

6. $\int \cot x \, dx = \log_e |\sin x|$

7. $\int \frac{dx}{\cos^2 x} = \tan x$

8. $\int \frac{dx}{\sin^2 x} = -\cot x$

9. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} \qquad (a \neq 0)$

10. $\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a} = \frac{1}{2a} \log_e \frac{a + x}{a - x} \qquad (|x| < a)$

11. $\int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \coth^{-1} \frac{x}{a} = \frac{1}{2a} \log_e \frac{x - a}{x + a} \qquad (|x| > a)$

12. $\int e^x \, dx = e^x$

13. $\int a^x \, dx = \frac{a^x}{\log_e a} \qquad (a > 0, a \neq 1)$

14. $\int \sinh x \, dx = \cosh x$

15. $\int \cosh x \, dx = \sinh x$

16. $\int \tanh x \, dx = \log_e \cosh x$

17. $\int \coth x \, dx = \log_e |\sinh x|$

18. $\int \frac{dx}{\cosh^2 x} = \tanh x$

19. $\int \frac{dx}{\sinh^2 x} = -\coth x$

20. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} \qquad (a \neq 0)$

21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a} = \log_e |x + \sqrt{a^2 + x^2}| + C_1$

22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} = \log_e |x + \sqrt{x^2 - a^2}| + C_1$

(a) *Integrals containing* $ax + b$ $\quad (a \neq 0)$

1. $$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} \qquad (n \neq -1)$$

2. $$\int \frac{dx}{ax+b} = \frac{1}{a}\log_e(ax+b)$$

3. $$\int x(ax+b)^n\,dx = \frac{1}{a^2(n+2)}(ax+b)^{n+2} - \frac{b}{a^2(n+1)}(ax+b)^{n+1} \qquad (n \neq -1, -2)$$

4. $$\int x^m(ax+b)^n\,dx$$
$$= \frac{1}{a(m+n+1)}\left[x^m(ax+b)^{n+1} - mb\int x^{m-1}(ax+b)^n\,dx\right]$$
$$= \frac{1}{m+n+1}\left[x^{m+1}(ax+b)^n + nb\int x^m(ax+b)^{n-1}\,dx\right]$$
$$(m > 0,\ m+n+1 \neq 0)$$

5. $$\int \frac{x\,dx}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\log_e(ax+b)$$

6. $$\int \frac{x\,dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\log_e(ax+b)$$

7. $$\int \frac{x\,dx}{(ax+b)^3} = \frac{b}{2a^2(ax+b)^2} - \frac{1}{a^2(ax+b)}$$

8. $$\int \frac{x\,dx}{(ax+b)^n} = \frac{1}{a^2}\left[\frac{b}{(n-1)(ax+b)^{n-1}} - \frac{1}{(n-2)(ax+b)^{n-2}}\right]$$
$$(n \neq 1, 2)$$

9. $$\int \frac{x^2\,dx}{ax+b} = \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b(ax+b) + b^2 \log_e (ax+b)\right]$$

10. $$\int \frac{x^2\,dx}{(ax+b)^2} = \frac{1}{a^3}\left[(ax+b) - 2b \log_e (ax+b) - \frac{b^2}{ax+b}\right]$$

11. $$\int \frac{x^2\,dx}{(ax+b)^3} = \frac{1}{a^3}\left[\log_e (ax+b) + \frac{2b}{ax+b} - \frac{b^2}{2(ax+b)^2}\right]$$

12. $$\int x^2(ax+b)^n\,dx$$
$$= \frac{1}{a^3}\left[\frac{(ax+b)^{n+3}}{n+3} - 2b\,\frac{(ax+b)^{n+2}}{n+2} + b^2\,\frac{(ax+b)^{n+1}}{n+1}\right]$$
$$(n \neq -1, -2, -3)$$

13. $$\int \frac{dx}{x(ax+b)} = \frac{1}{b} \log_e \frac{x}{ax+b}$$

14. $$\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} - \frac{1}{b^2} \log_e \frac{ax+b}{x}$$

15. $$\int \frac{dx}{x(ax+b)^3} = \frac{1}{b^3}\left[\frac{1}{2}\left(\frac{ax+2b}{ax+b}\right)^2 + \log_e \frac{x}{ax+b}\right]$$

16. $$\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2} \log_e \frac{ax+b}{x}$$

17. $$\int \frac{dx}{x^2(ax+b)^2} = -\frac{b+2ax}{b^2x(ax+b)} + \frac{2a}{b^3} \log_e \frac{ax+b}{x}$$

18. $$\int \frac{dx}{x^3(ax+b)} = \frac{2ax-b}{2b^2x^2} + \frac{a^2}{b^3} \log_e \frac{x}{ax+b}$$

Let $ax + b \equiv X$ $(a \neq 0)$. Then

19. $$\int \frac{x^3\,dx}{X} = \frac{1}{a^4}\left(\frac{X^3}{3} - \frac{3bX^2}{2} + 3b^2X - b^3 \log_e X\right)$$

20. $$\int \frac{x^3\,dx}{X^2} = \frac{1}{a^4}\left(\frac{X^2}{2} - 3bX + 3b^2 \log_e X + \frac{b^3}{X}\right)$$

21. $$\int \frac{x^3\,dx}{X^3} = \frac{1}{a^4}\left(X - 3b \log_e X - \frac{3b^2}{X} + \frac{b^3}{2X^2}\right)$$

22. $$\int \frac{x^3\,dx}{X^4} = \frac{1}{a^4}\left(\log_e X + \frac{3b}{X} - \frac{3b^2}{2X^2} + \frac{b^3}{3X^3}\right)$$

23. $$\int \frac{x^3\,dx}{X^n}$$
$$= \frac{1}{a^4}\left[\frac{-1}{(n-4)X^{n-4}} + \frac{3b}{(n-3)X^{n-3}} - \frac{3b^2}{(n-2)X^{n-2}} + \frac{b^3}{(n-1)X^{n-1}}\right]$$
$$(n \neq 1, 2, 3, 4)$$

24. $$\int \frac{dx}{xX^n} = -\frac{1}{b^n}\left[\log_e \frac{X}{x} - \sum_{k=1}^{n-1}\binom{n-1}{k}\frac{(-a)^k x^k}{kX^k}\right], \quad (n \geq 1)$$

25. $$\int \frac{dx}{x^2X^3} = -a\left[\frac{1}{2b^2X^2} + \frac{2}{b^3X} + \frac{1}{ab^3x} - \frac{3}{b^4}\log_e \frac{X}{x}\right]$$

26. $$\int \frac{dx}{x^2X^n} = -\frac{1}{b^{n+1}}\left[-\sum_{k=2}^{n}\binom{n}{k}\frac{(-a)^k x^{k-1}}{(k-1)X^{k-1}} + \frac{X}{x} - na\log_e \frac{X}{x}\right]$$
$$(n \geq 2)$$

27. $$\int \frac{dx}{x^3X^2} = -\frac{1}{b^4}\left[3a^2\log_e \frac{X}{x} + \frac{a^3x}{X} + \frac{X^2}{2x^2} - \frac{3aX}{x}\right]$$

28. $$\int \frac{dx}{x^3X^3} = -\frac{1}{b^5}\left[6a^2\log_e \frac{X}{x} + \frac{4a^3x}{X} - \frac{a^4x^2}{2X^2} + \frac{X^2}{2x^2} - \frac{4aX}{x}\right]$$

29. $$\int \frac{dx}{x^3X^n} = -\frac{1}{b^{n+2}}\left[-\sum_{k=3}^{n+1}\binom{n+1}{k}\frac{(-a)^k x^{k-2}}{(k-2)X^{k-2}} + \frac{a^2X^2}{2x^2}\right.$$
$$\left. - \frac{(n+1)aX}{x} + \frac{n(n+1)a^2}{2}\log_e \frac{X}{x}\right] \quad (n \geq 3)$$

30. $$\int \frac{dx}{x^mX^n} = -\frac{1}{b^{m+n-1}}\sum_{k=0}^{m+n-2}\binom{m+n-2}{k}\frac{X^{m-k-1}(-a)^k}{(m-k-1)x^{m-k-1}}$$
$$\left[\text{terms with } (m-k-1) = 0 \text{ are replaced by } \binom{m+n-2}{m-1}(-a)^{m-1}\log_e \frac{X}{x}\right]$$

(b) *Integrals containing* $ax + b$ *and* $cx + d$ $\quad (a \neq 0, c \neq 0)$

31. $$\int \frac{ax+b}{cx+d}\,dx = \frac{a}{c}x + \frac{bc-ad}{c^2}\log_e (cx+d)$$

32. $$\int \frac{dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\log_e \frac{cx+d}{ax+b} \qquad (bc-ad \neq 0)$$

33. $$\int \frac{x\,dx}{(ax+b)(cx+d)}$$

$$= \frac{1}{bc-ad}\left[\frac{b}{a}\log(ax+b) - \frac{d}{c}\log_e(cx+d)\right] \qquad (bc-ad \neq 0)$$

34. $$\int \frac{dx}{(ax+b)^2(cx+d)}$$

$$= \frac{1}{bc-ad}\left[\frac{1}{ax+b} + \frac{c}{bc-ad}\log_e \frac{cx+d}{ax+b}\right] \qquad (bc-ad \neq 0)$$

(c) *Integrals containing* $a+x$ *and* $b+x$ $\qquad (a \neq b)$

35. $$\int \frac{x\,dx}{(a+x)(b+x)^2} = \frac{b}{(a-b)(b+x)} - \frac{a}{(a-b)^2}\log_e \frac{a+x}{b+x}$$

36. $$\int \frac{x^2\,dx}{(a+x)(b+x)^2} = \frac{b^2}{(b-a)(b+x)} + \frac{a^2}{(b-a)^2}\log_e(a+x) + \frac{b^2-2ab}{(b-a)^2}\log_e(b+x)$$

37. $$\int \frac{dx}{(a+x)^2(b+x)^2} = \frac{-1}{(a-b)^2}\left(\frac{1}{a+x} + \frac{1}{b+x}\right) + \frac{2}{(a-b)^3}\log_e \frac{a+x}{b+x}$$

38. $$\int \frac{x\,dx}{(a+x)^2(b+x)^2} = \frac{1}{(a-b)^2}\left(\frac{a}{a+x} + \frac{b}{b+x}\right) + \frac{a+b}{(a-b)^3}\log_e \frac{a+x}{b+x}$$

39. $$\int \frac{x^2\,dx}{(a+x)^2(b+x)^2} = \frac{-1}{(a-b)^2}\left(\frac{a^2}{a+x} + \frac{b^2}{b+x}\right) + \frac{2ab}{(a-b)^3}\log_e \frac{a+x}{b+x}$$

(d) *Integrals containing* $ax^2 + bx + c \qquad (a \neq 0)$

40. $$\int \frac{dx}{ax^2+bx+c}$$

$$= \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}} \log_e \dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} & (b^2 > 4ac) \\ \dfrac{2}{\sqrt{4ac-b^2}} \arctan \dfrac{2ax+b}{\sqrt{4ac-b^2}} & (b^2 < 4ac) \\ -\dfrac{2}{2ax+b} & (b^2 = 4ac) \end{cases}$$

In numbers 41 through 47, let $b^2 - 4ac \neq 0$.

41. $$\int \frac{dx}{(ax^2+bx+c)^2} = \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \frac{2a}{(4ac-b^2)} \int \frac{dx}{ax^2+bx+c}$$

42. $$\int \frac{dx}{(ax^2+bx+c)^{n+1}} = \frac{2ax+b}{n(4ac-b^2)(ax^2+bx+c)^n} + \frac{2(2n-1)a}{n(4ac-b^2)} \int \frac{dx}{(ax^2+bx+c)^n}$$

43. $$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a} \log_e (ax^2+bx+c) - \frac{b}{2a} \int \frac{dx}{ax^2+bx+c}$$

44. $$\int \frac{x^2\,dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2} \log_e (ax^2+bx+c) + \frac{b^2-2ac}{2a^2} \int \frac{dx}{ax^2+bx+c}$$

45. $$\int \frac{x^n\,dx}{ax^2+bx+c} = \frac{x^{n-1}}{(n-1)a} - \frac{c}{a} \int \frac{x^{n-2}\,dx}{ax^2+bx+c} - \frac{b}{a} \int \frac{x^{n-1}\,dx}{ax^2+bx+c} \qquad (n \neq 1)$$

46. $$\int \frac{x\,dx}{(ax^2+bx+c)^{n+1}} = \frac{-(2c+bx)}{n(4ac-b^2)(ax^2+bx+c)^n} - \frac{b(2n-1)}{n(4ac-b^2)} \int \frac{dx}{(ax^2+bx+c)^n}$$

47. $$\int \frac{x^m\,dx}{(ax^2+bx+c)^{n+1}} = -\frac{x^{m-1}}{a(2n-m+1)(ax^2+bx+c)^n} - \frac{n-m+1}{2n-m+1}\cdot\frac{b}{a}\int \frac{x^{m-1}\,dx}{(ax^2+bx+c)^{n+1}} + \frac{m-1}{2n-m+1}\cdot\frac{c}{a}\int \frac{x^{m-2}\,dx}{(ax^2+bx+c)^{n+1}} \qquad (m \neq 2n+1)$$

Let $ax^2+bx+c \equiv X$ $(a \neq 0)$. Then

48. $$\int \frac{x^{2n-1}\,dx}{X^n} = \frac{1}{a}\int \frac{x^{2n-3}\,dx}{X^{n-1}} - \frac{c}{a}\int \frac{x^{2n-3}\,dx}{X^n} - \frac{b}{a}\int \frac{x^{2n-2}\,dx}{X^n}$$

49. $$\int \frac{dx}{xX} = \frac{1}{2c}\log_e \frac{x^2}{X} - \frac{b}{2c}\int \frac{dx}{X}$$

50. $$\int \frac{dx}{xX^n} = \frac{1}{2c(n-1)X^{n-1}} - \frac{b}{2c}\int \frac{dx}{X^n} + \frac{1}{c}\int \frac{dx}{xX^{n-1}}$$

51. $$\int \frac{dx}{x^2X} = \frac{b}{2c^2}\log_e \frac{X}{x^2} - \frac{1}{cx} + \left(\frac{b^2}{2c^2} - \frac{a}{c}\right)\int \frac{dx}{X}$$

52. $$\int \frac{dx}{x^mX^n} = -\frac{1}{(m-1)cx^{m-1}X^{n-1}} - \frac{(2n+m-3)a}{(m-1)c}\int \frac{dx}{x^{m-2}X^n} - \frac{(n+m-2)b}{(m-1)c}\int \frac{dx}{x^{m-1}X^n} \qquad (m > 1)$$

53. $$\int \frac{dx}{(fx+g)X} = \frac{1}{2(cf^2-gbf+g^2a)}\left[f\log_e \frac{(fx+g)^2}{X}\right] + \frac{2ga-bf}{2(cf^2-gbf+g^2a)}\int \frac{dx}{X}$$

(e) *Integrals containing* $a^2 \pm x^2$, *with*

$X \equiv a^2+x^2 \qquad Y \equiv \arctan\dfrac{x}{a}$

or

$$X \equiv a^2-x^2 \qquad Y \equiv \tanh^{-1}\frac{x}{a} \equiv \begin{cases} \dfrac{1}{2}\log_e \dfrac{a+x}{a-x} & (|x| < a) \\ \dfrac{1}{2}\log_e \dfrac{x+a}{x-a} & (|x| > a) \end{cases}$$

Where $\pm$ or $\mp$ appears in a formula, the upper sign refers to $X \equiv a^2 + x^2$ and the lower sign to $X \equiv a^2 - x^2$ $(a \neq 0)$.

54. $\displaystyle\int \frac{dx}{X} = \frac{1}{a} Y$

55. $\displaystyle\int \frac{dx}{X^2} = \frac{x}{2a^2X} + \frac{1}{2a^3} Y$

56. $\displaystyle\int \frac{dx}{X^3} = \frac{x}{4a^2X^2} + \frac{3x}{8a^4X} + \frac{3}{8a^5} Y$

57. $\displaystyle\int \frac{dx}{X^{n+1}} = \frac{x}{2na^2X^n} + \frac{2n-1}{2na^2} \int \frac{dx}{X^n}$

58. $\displaystyle\int \frac{x\,dx}{X} = \pm \frac{1}{2} \log_e X$

59. $\displaystyle\int \frac{x\,dx}{X^2} = \mp \frac{1}{2X}$

60. $\displaystyle\int \frac{x\,dx}{X^3} = \mp \frac{1}{4X^2}$

61. $\displaystyle\int \frac{x\,dx}{X^{n+1}} = \mp \frac{1}{2nX^n} \qquad (n \neq 0)$

62. $\displaystyle\int \frac{x^2\,dx}{X} = \pm x \mp aY$

63. $\displaystyle\int \frac{x^2\,dx}{X^2} = \mp \frac{x}{2X} \pm \frac{1}{2a} Y$

64. $\displaystyle\int \frac{x^2\,dx}{X^3} = \mp \frac{x}{4X^2} \pm \frac{x}{8a^2X} \pm \frac{1}{8a^3} Y$

65. $\displaystyle\int \frac{x^2\,dx}{X^{n+1}} = \mp \frac{x}{2nX^n} \pm \frac{1}{2n} \int \frac{dx}{X^n} \qquad (n \neq 0)$

66. $\displaystyle\int \frac{x^3\,dx}{X} = \pm \frac{x^2}{2} - \frac{a^2}{2} \log_e X$

67. $\displaystyle\int \frac{x^3\,dx}{X^2} = \frac{a^2}{2X} + \frac{1}{2} \log_e X$

68. $\displaystyle\int \frac{x^3\,dx}{X^3} = -\frac{1}{2X} + \frac{a^2}{4X^2}$

69. $\int \frac{x^3\,dx}{X^{n+1}} = -\frac{1}{2(n-1)X^{n-1}} + \frac{a^2}{2nX^n} \qquad (n > 1)$

70. $\int \frac{dx}{xX} = \frac{1}{2a^2}\log_e \frac{x^2}{X}$

71. $\int \frac{dx}{xX^2} = \frac{1}{2a^2X} + \frac{1}{2a^4}\log_e \frac{x^2}{X}$

72. $\int \frac{dx}{xX^3} = \frac{1}{4a^2X^2} + \frac{1}{2a^4X} + \frac{1}{2a^6}\log_e \frac{x^2}{X}$

73. $\int \frac{dx}{x^2X} = -\frac{1}{a^2x} \mp \frac{1}{a^3}Y$

74. $\int \frac{dx}{x^2X^2} = -\frac{1}{a^4x} \mp \frac{x}{2a^4X} \mp \frac{3}{2a^5}Y$

75. $\int \frac{dx}{x^2X^3} = -\frac{1}{a^6x} \mp \frac{x}{4a^4X^2} \mp \frac{7x}{8a^6X} \mp \frac{15}{8a^7}Y$

76. $\int \frac{dx}{x^3X} = -\frac{1}{2a^2x^2} \mp \frac{1}{2a^4}\log_e \frac{x^2}{X}$

77. $\int \frac{dx}{x^3X^2} = -\frac{1}{2a^4x^2} \mp \frac{1}{2a^4X} \mp \frac{1}{a^6}\log_e \frac{x^2}{X}$

78. $\int \frac{dx}{x^3X^3} = -\frac{1}{2a^6x^2} \mp \frac{1}{a^6X} \mp \frac{1}{4a^4X^2} \mp \frac{3}{2a^8}\log_e \frac{x^2}{X}$

79. $\int \frac{dx}{(b+cx)X} = \frac{1}{a^2c^2 \pm b^2}\left[c\log_e (b+cx) - \frac{c}{2}\log_e X \pm \frac{b}{a}Y\right]$

(f) Integrals containing $a^3 \pm x^3$, with

$$X \equiv a^3 \pm x^3 \qquad (a \neq 0)$$

Where $\pm$ or $\mp$ appears in a formula, the upper sign refers to $X \equiv a^3 + x^3$ and the lower sign to $X \equiv a^3 - x^3$.

80. $\int \frac{dx}{X} = \pm\frac{1}{6a^2}\log_e \frac{(a \pm x)^2}{a^2 \mp ax + x^2} + \frac{1}{a^2\sqrt{3}}\arctan\frac{2x \mp a}{a\sqrt{3}}$

81. $\int \frac{dx}{X^2} = \frac{x}{3a^3X} + \frac{2}{3a^3}\int \frac{dx}{X}$

82. $\int \frac{x\,dx}{X} = \frac{1}{6a}\log_e \frac{a^2 \mp ax + x^2}{(a \pm x)^2} \pm \frac{1}{a\sqrt{3}} \arctan \frac{2x \mp a}{a\sqrt{3}}$

83. $\int \frac{x\,dx}{X^2} = \frac{x^2}{3a^3X} + \frac{1}{3a^3}\int \frac{x\,dx}{X}$

84. $\int \frac{x^2\,dx}{X} = \pm\frac{1}{3}\log_e X$

85. $\int \frac{x^2\,dx}{X^2} = \mp\frac{1}{3X}$

86. $\int \frac{x^3\,dx}{X} = \pm x \mp a^3 \int \frac{dx}{X}$

87. $\int \frac{x^3\,dx}{X^2} = \mp\frac{x}{3X} \pm \frac{1}{3}\int \frac{dx}{X}$

88. $\int \frac{dx}{xX} = \frac{1}{3a^3}\log_e \frac{x^3}{X}$

89. $\int \frac{dx}{xX^2} = \frac{1}{3a^3X} + \frac{1}{3a^6}\log_e \frac{x^3}{X}$

90. $\int \frac{dx}{x^2X} = -\frac{1}{a^3x} \mp \frac{1}{a^3}\int \frac{x\,dx}{X}$

91. $\int \frac{dx}{x^2X^2} = -\frac{1}{a^6x} \mp \frac{x^2}{3a^6X} \mp \frac{4}{3a^6}\int \frac{x\,dx}{X}$

92. $\int \frac{dx}{x^3X} = -\frac{1}{2a^3x^2} \mp \frac{1}{a^3}\int \frac{dx}{X}$

93. $\int \frac{dx}{x^3X^2} = -\frac{1}{2a^6x^2} \mp \frac{x}{3a^6X} \mp \frac{5}{3a^6}\int \frac{dx}{X}$

(g) *Integrals containing* $a^4 \pm x^4$ $\quad (a \neq 0)$

94. $\int \frac{dx}{a^4 + x^4} = \frac{1}{4a^3\sqrt{2}}\log_e \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a^3\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2 - x^2}$

95. $\int \frac{x\,dx}{a^4 + x^4} = \frac{1}{2a^2} \arctan \frac{x^2}{a^2}$

96. $\int \frac{x^2\,dx}{a^4 + x^4} = -\frac{1}{4a\sqrt{2}}\log_e \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2 - x^2}$

97. $\int \frac{x^3\,dx}{a^4+x^4} = \frac{1}{4}\log_e(a^4+x^4)$

98. $\int \frac{dx}{a^4-x^4} = \frac{1}{4a^3}\log_e\frac{a+x}{a-x} + \frac{1}{2a^3}\arctan\frac{x}{a}$

99. $\int \frac{x\,dx}{a^4-x^4} = \frac{1}{4a^3}\log_e\frac{a^2+x^2}{a^2-x^2}$

100. $\int \frac{x^2\,dx}{a^4-x^4} = \frac{1}{4a}\log_e\frac{a+x}{a-x} - \frac{1}{2a}\arctan\frac{x}{a}$

101. $\int \frac{x^3\,dx}{a^4-x^4} = -\frac{1}{4}\log_e(a^4-x^4)$

(h) *Integrals containing* $\sqrt{x}$ *and* a^2+b^2x $\quad (a, b \neq 0)$

102. $\int \frac{\sqrt{x}\,dx}{a^2+b^2x} = \frac{2\sqrt{x}}{b^2} - \frac{2a}{b^3}\arctan\frac{b\sqrt{x}}{a}$

103. $\int \frac{x\sqrt{x}\,dx}{a^2+b^2x} = \frac{2x\sqrt{x}}{3b^2} - \frac{2a^2\sqrt{x}}{b^4} + \frac{2a^3}{b^5}\arctan\frac{b\sqrt{x}}{a}$

104. $\int \frac{\sqrt{x}\,dx}{(a^2+b^2x)^2} = -\frac{\sqrt{x}}{b^2(a^2+b^2x)} + \frac{1}{ab^3}\arctan\frac{b\sqrt{x}}{a}$

105. $\int \frac{x\sqrt{x}\,dx}{(a^2+b^2x)^2} = \frac{2b^2x\sqrt{x}+3a^2\sqrt{x}}{b^4(a^2+b^2x)} - \frac{3a}{b^5}\arctan\frac{b\sqrt{x}}{a}$

106. $\int \frac{dx}{(a^2+b^2x)\sqrt{x}} = \frac{2}{ab}\arctan\frac{b\sqrt{x}}{a}$

107. $\int \frac{dx}{(a^2+b^2x)x\sqrt{x}} = -\frac{2}{a^2\sqrt{x}} - \frac{2b}{a^3}\arctan\frac{b\sqrt{x}}{a}$

108. $\int \frac{dx}{(a^2+b^2x)^2\sqrt{x}} = \frac{\sqrt{x}}{a^2(a^2+b^2x)} + \frac{1}{a^3b}\arctan\frac{b\sqrt{x}}{a}$

(i) *Integrals containing* $\sqrt{x}$ *and* $a^2-b^2x > 0$ $\quad (a, b \neq 0)$

109. $\int \frac{\sqrt{x}\,dx}{a^2-b^2x} = -\frac{2\sqrt{x}}{b^2} + \frac{a}{b^3}\log_e\frac{a+b\sqrt{x}}{a-b\sqrt{x}}$

110. $\int \frac{x\sqrt{x}\,dx}{a^2-b^2x} = -\frac{2x\sqrt{x}}{3b^2} - \frac{2a^2\sqrt{x}}{b^4} + \frac{a^3}{b^5}\log_e\frac{a+b\sqrt{x}}{a-b\sqrt{x}}$

111. $$\int \frac{\sqrt{x}\,dx}{(a^2-b^2x)^2} = \frac{\sqrt{x}}{b^2(a^2-b^2x)} - \frac{1}{2ab^3}\log_e\frac{a+b\sqrt{x}}{a-b\sqrt{x}}$$

112. $$\int \frac{x\sqrt{x}\,dx}{(a^2-b^2x)^2} = \frac{-2b^2x\sqrt{x}+3a^2\sqrt{x}}{b^4(a^2-b^2x)} - \frac{3a}{2b^5}\log_e\frac{a+b\sqrt{x}}{a-b\sqrt{x}}$$

113. $$\int \frac{dx}{(a^2-b^2x)\sqrt{x}} = \frac{1}{ab}\log_e\frac{a+b\sqrt{x}}{a-b\sqrt{x}}$$

114. $$\int \frac{dx}{(a^2-b^2x)x\sqrt{x}} = -\frac{2}{a^2\sqrt{x}} + \frac{b}{a^3}\log_e\frac{a+b\sqrt{x}}{a-b\sqrt{x}}$$

115. $$\int \frac{dx}{(a^2-b^2x)^2\sqrt{x}} = \frac{\sqrt{x}}{a^2(a^2-b^2x)} + \frac{1}{2a^3b}\log_e\frac{a+b\sqrt{x}}{a-b\sqrt{x}}$$

116. $$\int \frac{dx}{(a^2-b^2x)^2x\sqrt{x}} = -\frac{2}{a^2(a^2-b^2x)\sqrt{x}} + \frac{3b^2\sqrt{x}}{a^4(a^2-b^2x)} + \frac{3b}{2a^5}\log_e\frac{a+b\sqrt{x}}{a-b\sqrt{x}}$$

(j) *Other integrals containing* $\sqrt{x}$ $\quad (a > \sqrt{x} > 0)$

117. $$\int \frac{\sqrt{x}\,dx}{a^4+x^2} = -\frac{1}{2a\sqrt{2}}\log_e\frac{x+a\sqrt{2x}+a^2}{x-a\sqrt{2x}+a^2} + \frac{1}{a\sqrt{2}}\arctan\frac{a\sqrt{2x}}{a^2-x}$$

118. $$\int \frac{dx}{(a^4+x^2)\sqrt{x}} = \frac{1}{2a^3\sqrt{2}}\log_e\frac{x+a\sqrt{2x}+a^2}{x-a\sqrt{2x}+a^2} + \frac{1}{a^3\sqrt{2}}\arctan\frac{a\sqrt{2x}}{a^2-x}$$

119. $$\int \frac{\sqrt{x}\,dx}{a^4-x^2} = \frac{1}{2a}\log_e\frac{a+\sqrt{x}}{a-\sqrt{x}} - \frac{1}{a}\arctan\frac{\sqrt{x}}{a}$$

120. $$\int \frac{dx}{(a^4-x^2)\sqrt{x}} = \frac{1}{2a^3}\log_e\frac{a+\sqrt{x}}{a-\sqrt{x}} + \frac{1}{a^3}\arctan\frac{\sqrt{x}}{a}$$

(k) *Integrals containing* $\sqrt{ax+b}$, *with*

$X \equiv ax+b \qquad Y \equiv fx+g \qquad \Delta \equiv bf-ag \qquad (a \neq 0)$

121. $\int \sqrt{X}\,dx = \frac{2}{3a}\sqrt{X^3}$

122. $\int x\sqrt{X}\,dx = \frac{2(3ax - 2b)\sqrt{X^3}}{15a^2}$

123. $\int x^2\sqrt{X}\,dx = \frac{2(15a^2x^2 - 12abx + 8b^2)\sqrt{X^3}}{105a^3}$

124. $\int \frac{dx}{\sqrt{X}} = \frac{2\sqrt{X}}{a}$

125. $\int \frac{x\,dx}{\sqrt{X}} = \frac{2(ax - 2b)}{3a^2}\sqrt{X}$

126. $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{2(3a^2x^2 - 4abx + 8b^2)\sqrt{X}}{15a^3}$

127. $\int \frac{dx}{x\sqrt{X}} = \begin{cases} -\frac{2}{\sqrt{b}}\tanh^{-1}\sqrt{\frac{X}{b}} = \frac{1}{\sqrt{b}}\log_e\frac{\sqrt{X}-\sqrt{b}}{\sqrt{X}+\sqrt{b}} & (b > 0) \\ \frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{X}{-b}} & (b < 0) \end{cases}$

128. $\int \frac{\sqrt{X}}{x}\,dx = 2\sqrt{X} + b\int \frac{dx}{x\sqrt{X}}$

129. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{X}}$

130. $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{X}}$

131. $\int \frac{dx}{x^n\sqrt{X}} = -\frac{\sqrt{X}}{(n-1)bx^{n-1}} - \frac{(2n-3)a}{(2n-2)b}\int \frac{dx}{x^{n-1}\sqrt{X}} \qquad (n \neq 1)$

132. $\int \sqrt{X^3}\,dx = \frac{2\sqrt{X^5}}{5a}$

133. $\int x\sqrt{X^3}\,dx = \frac{2}{35a^2}(5\sqrt{X^7} - 7b\sqrt{X^5})$

134. $\int x^2\sqrt{X^3}\,dx = \frac{2}{a^3}\left(\frac{\sqrt{X^9}}{9} - \frac{2b\sqrt{X^7}}{7} + \frac{b^2\sqrt{X^5}}{5}\right)$

135. $$\int \frac{\sqrt{X^3}}{x}\,dx = \frac{2\sqrt{X^3}}{3} + 2b\sqrt{X} + b^2 \int \frac{dx}{x\sqrt{X}}$$

136. $$\int \frac{x\,dx}{\sqrt{X^3}} = \frac{2}{a^2}\left(\sqrt{X} + \frac{b}{\sqrt{X}}\right)$$

137. $$\int \frac{x^2\,dx}{\sqrt{X^3}} = \frac{2}{a^3}\left(\frac{\sqrt{X^3}}{3} - 2b\sqrt{X} - \frac{b^2}{\sqrt{X}}\right)$$

138. $$\int \frac{dx}{x\sqrt{X^3}} = \frac{2}{b\sqrt{X}} + \frac{1}{b}\int \frac{dx}{x\sqrt{X}} \qquad (b \neq 0)$$

139. $$\int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{bx\sqrt{X}} - \frac{3a}{b^2\sqrt{X}} - \frac{3a}{2b^2}\int \frac{dx}{x\sqrt{X}} \qquad (b \neq 0)$$

140. $$\int X^{\pm n/2}\,dx = \frac{2X^{(2\pm n)/2}}{a(2 \pm n)} \qquad (n \neq \mp 2)$$

141. $$\int xX^{\pm n/2}\,dx = \frac{2}{a^2}\left(\frac{X^{(4\pm n)/2}}{4 \pm n} - \frac{bX^{(2\pm n)/2}}{2 \pm n}\right) \qquad (n \neq \mp 2, \mp 4)$$

142. $$\int x^2X^{\pm n/2}\,dx = \frac{2}{a^3}\left(\frac{X^{(6\pm n)/2}}{6 \pm n} - \frac{2bX^{(4\pm n)/2}}{4 \pm n} + \frac{b^2X^{(2\pm n)/2}}{2 \pm n}\right)$$
$$(n \neq \mp 2, \mp 4, \mp 6)$$

143. $$\int \frac{X^{n/2}\,dx}{x} = \frac{2X^{n/2}}{n} + b\int \frac{X^{(n-2)/2}}{x}\,dx \qquad (n \neq 0)$$

144. $$\int \frac{dx}{xX^{n/2}} = \frac{2}{(n-2)bX^{(n-2)/2}} + \frac{1}{b}\int \frac{dx}{xX^{(n-2)/2}} \qquad (n \neq 2,\ b \neq 0)$$

145. $$\int \frac{dx}{x^2X^{n/2}} = -\frac{1}{bxX^{(n-2)/2}} - \frac{na}{2b}\int \frac{dx}{xX^{n/2}} \qquad (b \neq 0)$$

146. $$\int \frac{dx}{\sqrt{XY}} = \begin{cases} \dfrac{2}{\sqrt{-af}} \arctan\sqrt{-\dfrac{fX}{aY}} & (af < 0) \\ \dfrac{2}{\sqrt{af}} \tanh^{-1}\sqrt{\dfrac{fX}{aY}} = \dfrac{2}{\sqrt{af}} \log_e(\sqrt{aY} + \sqrt{fX}) + C_1 & \end{cases}$$
$$(af > 0)$$

147. $$\int \frac{x\,dx}{\sqrt{XY}} = \frac{\sqrt{XY}}{af} - \frac{ag + bf}{2af}\int \frac{dx}{\sqrt{XY}}$$

148. $$\int \frac{dx}{\sqrt{X}\sqrt{Y^3}} = -\frac{2\sqrt{X}}{\Delta\sqrt{Y}}$$

149. $$\int \frac{dx}{Y\sqrt{X}} = \begin{cases} \dfrac{2}{\sqrt{-f\Delta}} \arctan \dfrac{f\sqrt{X}}{\sqrt{-f\Delta}} & (f\Delta < 0) \\ \dfrac{1}{\sqrt{f\Delta}} \log_e \dfrac{f\sqrt{X} - \sqrt{f\Delta}}{f\sqrt{X} + \sqrt{f\Delta}} & (f\Delta > 0) \end{cases}$$

150. $$\int \sqrt{XY}\, dx = \frac{\Delta + 2aY}{4af}\sqrt{XY} - \frac{\Delta^2}{8af}\int \frac{dx}{\sqrt{XY}}$$

151. $$\int \sqrt{\frac{Y}{X}}\, dx = \frac{1}{a}\sqrt{XY} - \frac{\Delta}{2a}\int \frac{dx}{\sqrt{XY}}$$

152. $$\int \frac{\sqrt{X}\, dx}{Y} = \frac{2\sqrt{X}}{f} + \frac{\Delta}{f}\int \frac{dx}{Y\sqrt{X}} \qquad (f \neq 0)$$

153. $$\int \frac{Y^n\, dx}{\sqrt{X}} = \frac{2}{(2n+1)a}\left(\sqrt{X}Y^n - n\Delta \int \frac{Y^{n-1}\, dx}{\sqrt{X}}\right)$$

154. $$\int \frac{dx}{\sqrt{X}Y^n} = -\frac{1}{(n-1)\Delta}\left\{\frac{\sqrt{X}}{Y^{n-1}} + \left(n - \frac{3}{2}\right)a \int \frac{dx}{\sqrt{X}Y^{n-1}}\right\}$$

$(\Delta \neq 0, n \neq 1)$

155. $$\int \sqrt{X}Y^n\, dx = \frac{1}{(2n+3)f}\left(2\sqrt{X}Y^{n+1} + \Delta \int \frac{Y^n\, dx}{\sqrt{X}}\right) \qquad (f \neq 0)$$

156. $$\int \frac{\sqrt{X}\, dx}{Y^n} = \frac{1}{(n-1)f}\left(-\frac{\sqrt{X}}{Y^{n-1}} + \frac{a}{2}\int \frac{dx}{\sqrt{X}Y^{n-1}}\right) \qquad (f \neq 0, n \neq 1)$$

(*l*) *Integrals containing* $\sqrt{a^2 - x^2}$, *with*

$X \equiv a^2 - x^2 \qquad (a > 0)$

157. $$\int \sqrt{X}\, dx = \frac{1}{2}\left(x\sqrt{X} + a^2 \arcsin \frac{x}{a}\right)$$

158. $$\int x\sqrt{X}\, dx = -\frac{1}{3}\sqrt{X^3}$$

159. $\int x^2\sqrt{X}\,dx = -\frac{x}{4}\sqrt{X^3} + \frac{a^2}{8}\left(x\sqrt{X} + a^2 \arcsin\frac{x}{a}\right)$

160. $\int x^3\sqrt{X}\,dx = \frac{\sqrt{X^5}}{5} - a^2\frac{\sqrt{X^3}}{3}$

161. $\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\log_e\frac{a+\sqrt{X}}{x}$

162. $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} - \arcsin\frac{x}{a}$

163. $\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\log_e\frac{a+\sqrt{X}}{x}$

164. $\int \frac{dx}{\sqrt{X}} = \arcsin\frac{x}{a}$

165. $\int \frac{x\,dx}{\sqrt{X}} = -\sqrt{X}$

166. $\int \frac{x^2\,dx}{\sqrt{X}} = -\frac{x}{2}\sqrt{X} + \frac{a^2}{2}\arcsin\frac{x}{a}$

167. $\int \frac{x^3\,dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} - a^2\sqrt{X}$

168. $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\log_e\frac{a+\sqrt{X}}{x}$

169. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x}$

170. $\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} - \frac{1}{2a^3}\log_e\frac{a+\sqrt{X}}{x}$

171. $\int \sqrt{X^3}\,dx = \frac{1}{4}\left(x\sqrt{X^3} + \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\arcsin\frac{x}{a}\right)$

172. $\int x\sqrt{X^3}\,dx = -\frac{1}{5}\sqrt{X^5}$

173. $\int x^2\sqrt{X^3}\,dx = -\frac{x\sqrt{X^5}}{6} + \frac{a^2x\sqrt{X^3}}{24} + \frac{a^4x\sqrt{X}}{16} + \frac{a^6}{16}\arcsin\frac{x}{a}$

174. $\int x^3\sqrt{X^3}\,dx = \frac{\sqrt{X^7}}{7} - \frac{a^2\sqrt{X^5}}{5}$

175. $\int \frac{\sqrt{X^3}}{x}\,dx = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X} - a^3 \log_e \frac{a+\sqrt{X}}{x}$

176. $\int \frac{\sqrt{X^3}}{x^2}\,dx = -\frac{\sqrt{X^3}}{x} - \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2 \arcsin\frac{x}{a}$

177. $\int \frac{\sqrt{X^3}}{x^3}\,dx = -\frac{\sqrt{X^3}}{2x^2} - \frac{3\sqrt{X}}{2} + \frac{3a}{2}\log_e \frac{a+\sqrt{X}}{x}$

178. $\int \frac{dx}{\sqrt{X^3}} = \frac{x}{a^2\sqrt{X}}$

179. $\int \frac{x\,dx}{\sqrt{X^3}} = \frac{1}{\sqrt{X}}$

180. $\int \frac{x^2\,dx}{\sqrt{X^3}} = \frac{x}{\sqrt{X}} - \arcsin\frac{x}{a}$

181. $\int \frac{x^3\,dx}{\sqrt{X^3}} = \sqrt{X} + \frac{a^2}{\sqrt{X}}$

182. $\int \frac{dx}{x\sqrt{X^3}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\log_e \frac{a+\sqrt{X}}{x}$

183. $\int \frac{dx}{x^2\sqrt{X^3}} = \frac{1}{a^4}\left(-\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}}\right)$

184. $\int \frac{dx}{x^3\sqrt{X^3}} = -\frac{1}{2a^2x^2\sqrt{X}} + \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\log_e \frac{a+\sqrt{X}}{x}$

(*m*) *Integrals containing* $\sqrt{x^2+a^2}$, *with*

$X \equiv x^2 + a^2 \qquad (a > 0)$

185. $\int \sqrt{X}\,dx = \frac{1}{2}\left(x\sqrt{X} + a^2 \sinh^{-1}\frac{x}{a}\right)$

$= \frac{1}{2}\left[x\sqrt{X} + a^2 \log_e (x + \sqrt{X})\right] + C_1$

186. $\int x\sqrt{X}\,dx = \frac{1}{3}\sqrt{X^3}$

187. $\int x^2\sqrt{X}\,dx = \frac{x}{4}\sqrt{X^3} - \frac{a^2}{8}\left(x\sqrt{X} + a^2\sinh^{-1}\frac{x}{a}\right)$

$= \frac{x}{4}\sqrt{X^3} - \frac{a^2}{8}\left[x\sqrt{X} + a^2\log_e(x + \sqrt{X})\right] + C_1$

188. $\int x^3\sqrt{X}\,dx = \frac{\sqrt{X^5}}{5} - \frac{a^2\sqrt{X^3}}{3}$

189. $\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\log_e\frac{a + \sqrt{X}}{x}$

190. $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \sinh^{-1}\frac{x}{a}$

$= -\frac{\sqrt{X}}{x} + \log_e(x + \sqrt{X}) + C_1$

191. $\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} - \frac{1}{2a}\log_e\frac{a + \sqrt{X}}{x}$

192. $\int \frac{dx}{\sqrt{X}} = \sinh^{-1}\frac{x}{a} = \log_e(x + \sqrt{X}) + C_1$

193. $\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X}$

194. $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} - \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

$= \frac{x}{2}\sqrt{X} - \frac{a^2}{2}\log_e(x + \sqrt{X}) + C_1$

195. $\int \frac{x^3\,dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} - a^2\sqrt{X}$

196. $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\log_e\frac{a + \sqrt{X}}{x}$

197. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x}$

198. $$\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\log_e \frac{a+\sqrt{X}}{x}$$

199. $$\int \sqrt{X^3}\,dx = \frac{1}{4}\left(x\sqrt{X^3} + \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\sinh^{-1}\frac{x}{a}\right)$$
$$= \frac{1}{4}\left[x\sqrt{X^3} + \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\log_e (x+\sqrt{X})\right] + C_1$$

200. $$\int x\sqrt{X^3}\,dx = \frac{1}{5}\sqrt{X^5}$$

201. $$\int x^2\sqrt{X^3}\,dx = \frac{x\sqrt{X^5}}{6} - \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16} - \frac{a^6}{16}\sinh^{-1}\frac{x}{a}$$
$$= \frac{x\sqrt{X^5}}{6} - \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16}$$
$$- \frac{a^6}{16}\log_e (x+\sqrt{X}) + C_1$$

202. $$\int x^3\sqrt{X^3}\,dx = \frac{\sqrt{X^7}}{7} - \frac{a^2\sqrt{X^5}}{5}$$

203. $$\int \frac{\sqrt{X^3}}{x}\,dx = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X} - a^3\log_e \frac{a+\sqrt{X}}{x}$$

204. $$\int \frac{\sqrt{X^3}}{x^2}\,dx = -\frac{\sqrt{X^3}}{x} + \frac{3}{2}x\sqrt{X} + \frac{3}{2}a^2\sinh^{-1}\frac{x}{a}$$
$$= -\frac{\sqrt{X^3}}{x} + \frac{3}{2}x\sqrt{X} + \frac{3}{2}a^2\log_e (x+\sqrt{X}) + C_1$$

205. $$\int \frac{\sqrt{X^3}}{x^3}\,dx = -\frac{\sqrt{X^3}}{2x^2} + \frac{3}{2}\sqrt{X} - \frac{3}{2}a\log_e\left(\frac{a+\sqrt{X}}{x}\right)$$

206. $$\int \frac{dx}{\sqrt{X^3}} = \frac{x}{a^2\sqrt{X}}$$

207. $$\int \frac{x\,dx}{\sqrt{X^3}} = -\frac{1}{\sqrt{X}}$$

208. $$\int \frac{x^2\,dx}{\sqrt{X^3}} = -\frac{x}{\sqrt{X}} + \sinh^{-1}\frac{x}{a}$$
$$= -\frac{x}{\sqrt{X}} + \log_e (x+\sqrt{X}) + C_1$$

209. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X^3}} = \sqrt{X} + \frac{a^2}{\sqrt{X}}$

210. $\displaystyle\int \frac{dx}{x\sqrt{X^3}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\log_e \frac{a+\sqrt{X}}{x}$

211. $\displaystyle\int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{a^4}\left(\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}}\right)$

212. $\displaystyle\int \frac{dx}{x^3\sqrt{X^3}} = -\frac{1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} + \frac{3}{2a^5}\log_e \frac{a+\sqrt{X}}{x}$

(n) *Integrals containing* $\sqrt{x^2 - a^2}$, *with*

$$X \equiv x^2 - a^2 \qquad (a > 0)$$

213. $\displaystyle\int \sqrt{X}\,dx = \frac{1}{2}\left(x\sqrt{X} - a^2\cosh^{-1}\frac{x}{a}\right)$

$$= \frac{1}{2}[x\sqrt{X} - a^2\log_e(x+\sqrt{X})] + C_1$$

214. $\displaystyle\int x\sqrt{X}\,dx = \frac{1}{3}\sqrt{X^3}$

215. $\displaystyle\int x^2\sqrt{X}\,dx = \frac{x}{4}\sqrt{X^3} + \frac{a^2}{8}\left(x\sqrt{X} - a^2\cosh^{-1}\frac{x}{a}\right)$

$$= \frac{x}{4}\sqrt{X^3} + \frac{a^2}{8}[x\sqrt{X} - a^2\log_e(x+\sqrt{X})] + C_1$$

216. $\displaystyle\int x^3\sqrt{X}\,dx = \frac{\sqrt{X^5}}{5} + \frac{a^2\sqrt{X^3}}{3}$

217. $\displaystyle\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\arccos\frac{a}{x}$

218. $\displaystyle\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \cosh^{-1}\frac{x}{a}$

$$= -\frac{\sqrt{X}}{x} + \log_e(x+\sqrt{X}) + C_1$$

219. $\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\arccos\frac{a}{x}$

220. $\int \frac{dx}{\sqrt{X}} = \cosh^{-1}\frac{x}{a} = \log_e(x + \sqrt{X}) + C_1$

221. $\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X}$

222. $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} + \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

$= \frac{x}{2}\sqrt{X} + \frac{a^2}{2}\log_e(x + \sqrt{X}) + C_1$

223. $\int \frac{x^3\,dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X}$

224. $\int \frac{dx}{x\sqrt{X}} = \frac{1}{a}\arccos\frac{a}{x}$

225. $\int \frac{dx}{x^2\sqrt{X}} = \frac{\sqrt{X}}{a^2x}$

226. $\int \frac{dx}{x^3\sqrt{X}} = \frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\arccos\frac{a}{x}$

227. $\int \sqrt{X^3}\,dx = \frac{1}{4}\left(x\sqrt{X^3} - \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\cosh^{-1}\frac{x}{a}\right)$

$= \frac{1}{4}\left[x\sqrt{X^3} - \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\log_e(x + \sqrt{X})\right] + C_1$

228. $\int x\sqrt{X^3}\,dx = \frac{1}{5}\sqrt{X^5}$

229. $\int x^2\sqrt{X^3}\,dx = \frac{x\sqrt{X^5}}{6} + \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16} + \frac{a^6}{16}\cosh^{-1}\frac{x}{a}$

$= \frac{x\sqrt{X^5}}{6} + \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16}$

$+ \frac{a^6}{16}\log_e(x + \sqrt{X}) + C_1$

230. $\int x^3\sqrt{X^3}\,dx \quad \frac{\sqrt{X^7}}{7} + \frac{a^2\sqrt{X^5}}{5}$

231. $\int \frac{\sqrt{X^3}}{x}\,dx = \frac{\sqrt{X^3}}{3} - a^2\sqrt{X} + a^3 \arccos\frac{a}{x}$

232. $\int \frac{\sqrt{X^3}}{x^2}\,dx = -\frac{\sqrt{X^3}}{2} + \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2 \cosh^{-1}\frac{x}{a}$

$= -\frac{\sqrt{X^3}}{2} + \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2 \log_e (x + \sqrt{X}) + C_1$

233. $\int \frac{\sqrt{X^3}}{x^3}\,dx = -\frac{\sqrt{X^3}}{2x^2} + \frac{3\sqrt{X}}{2} - \frac{3}{2}a \arccos\frac{a}{x}$

234. $\int \frac{dx}{\sqrt{X^3}} = -\frac{x}{a^2\sqrt{X}}$

235. $\int \frac{x\,dx}{\sqrt{X^3}} = -\frac{1}{\sqrt{X}}$

236. $\int \frac{x^2\,dx}{\sqrt{X^3}} = -\frac{x}{\sqrt{X}} + \cosh^{-1}\frac{x}{a}$

$= -\frac{x}{\sqrt{X}} + \log_e (x + \sqrt{X}) + C_1$

237. $\int \frac{x^3\,dx}{\sqrt{X^3}} = \sqrt{X} - \frac{a^2}{\sqrt{X}}$

238. $\int \frac{dx}{x\sqrt{X^3}} = -\frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\arccos\frac{a}{x}$

239. $\int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{a^4}\left(\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}}\right)$

240. $\int \frac{dx}{x^3\sqrt{X^3}} = \frac{1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\arccos\frac{a}{x}$

(o) *Integrals containing* $\sqrt{ax^2 + bx + c}$, *with*

$X \equiv ax^2 + bx + c$

$\Delta = 4ac - b^2 \qquad k = \frac{4a}{4ac - b^2}$ $\qquad (a \neq 0)$

241. $$\int \frac{dx}{\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{a}} \log_e (2\sqrt{aX} + 2ax + b) + C & (a > 0) \\ \frac{1}{\sqrt{a}} \sinh^{-1} \frac{2ax + b}{\sqrt{\Delta}} + C_1 & (a > 0, \Delta > 0) \\ \frac{1}{\sqrt{a}} \log_e (2ax + b) & (a > 0, \Delta = 0) \\ -\frac{1}{\sqrt{-a}} \arcsin \frac{2ax + b}{\sqrt{-\Delta}} & (a < 0, \Delta < 0) \end{cases}$$

242. $$\int \frac{dx}{X\sqrt{X}} = \frac{2(2ax + b)}{\Delta\sqrt{X}} \qquad (\Delta \neq 0)$$

243. $$\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2ax + b)}{3\Delta\sqrt{X}} \left(\frac{1}{X} + 2k\right) \qquad (\Delta \neq 0)$$

244. $$\int \frac{dx}{X^{(2n+1)/2}} = \frac{2(2ax + b)}{(2n - 1)\Delta X^{(2n-1)/2}} + \frac{2k(n - 1)}{2n - 1} \int \frac{dx}{X^{(2n-1)/2}}$$

$(\Delta \neq 0)$

245. $$\int \sqrt{X}\, dx = \frac{(2ax + b)\sqrt{X}}{4a} + \frac{1}{2k} \int \frac{dx}{\sqrt{X}}$$

246. $$\int X\sqrt{X}\, dx = \frac{(2ax + b)\sqrt{X}}{8a} \left(X + \frac{3}{2k}\right) + \frac{3}{8k^2} \int \frac{dx}{\sqrt{X}}$$

247. $$\int X^2\sqrt{X}\, dx = \frac{(2ax + b)\sqrt{X}}{12a} \left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2}\right) + \frac{5}{16k^3} \int \frac{dx}{\sqrt{X}}$$

248. $$\int X^{(2n+1)/2}\, dx = \frac{(2ax + b)X^{(2n+1)/2}}{4a(n + 1)} + \frac{2n + 1}{2k(n + 1)} \int X^{(2n-1)/2}\, dx$$

249. $$\int \frac{x\, dx}{\sqrt{X}} = \frac{\sqrt{X}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{X}}$$

250. $$\int \frac{x\, dx}{X\sqrt{X}} = -\frac{2(bx + 2c)}{\Delta\sqrt{X}} \qquad (\Delta \neq 0)$$

251. $$\int \frac{x\, dx}{X^{(2n+1)/2}} = -\frac{1}{(2n - 1)aX^{(2n-1)/2}} - \frac{b}{2a} \int \frac{dx}{X^{(2n+1)/2}}$$

252. $$\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2a} - \frac{3b}{4a^2}\right)\sqrt{X} + \frac{3b^2 - 4ac}{8a^2}\int \frac{dx}{\sqrt{X}}$$

253. $$\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(2b^2 - 4ac)x + 2bc}{a\Delta\sqrt{X}} + \frac{1}{a}\int \frac{dx}{\sqrt{X}} \qquad (\Delta \neq 0)$$

254. $$\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3a} - \frac{b(2ax + b)}{8a^2}\sqrt{X} - \frac{b}{4ak}\int \frac{dx}{\sqrt{X}}$$

255. $$\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5a} - \frac{b}{2a}\int X\sqrt{X}\,dx$$

256. $$\int xX^{(2n+1)/2}\,dx = \frac{X^{(2n+3)/2}}{(2n+3)a} - \frac{b}{2a}\int X^{(2n+1)/2}\,dx$$

257. $$\int x^2\sqrt{X}\,dx = \left(x - \frac{5b}{6a}\right)\frac{X\sqrt{X}}{4a} + \frac{5b^2 - 4ac}{16a^2}\int \sqrt{X}\,dx$$

258. $$\int \frac{dx}{x\sqrt{X}} = \begin{cases} -\dfrac{1}{\sqrt{c}}\log_e\left(\dfrac{2\sqrt{cX}}{x} + \dfrac{2c}{x} + b\right) + C & (c > 0) \\ -\dfrac{1}{\sqrt{c}}\sinh^{-1}\dfrac{bx + 2c}{x\sqrt{\Delta}} + C_1 & (c > 0, \Delta > 0) \\ -\dfrac{1}{\sqrt{c}}\log_e\dfrac{bx + 2c}{x} & (c > 0, \Delta = 0) \\ \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx + 2c}{x\sqrt{-\Delta}} & (c < 0, \Delta < 0) \end{cases}$$

259. $$\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{cx} - \frac{b}{2c}\int \frac{dx}{x\sqrt{X}}$$

260. $$\int \frac{\sqrt{X}\,dx}{x} = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + c\int \frac{dx}{x\sqrt{X}}$$

261. $$\int \frac{\sqrt{X}\,dx}{x^2} = -\frac{\sqrt{X}}{x} + a\int \frac{dx}{\sqrt{X}} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}}$$

262. $$\int \frac{X^{(2n+1)/2}}{x}\,dx = \frac{X^{(2n+1)/2}}{2n+1} + \frac{b}{2}\int X^{(2n-1)/2}\,dx + c\int \frac{X^{(2n-1)/2}}{x}\,dx$$

(p) *Other irrational forms* $(a > 0, b \neq 0)$

263. $$\int \frac{dx}{x\sqrt{ax^2 + bx}} = -\frac{2}{bx}\sqrt{ax^2 + bx}$$

264. $$\int \frac{dx}{\sqrt{2ax - x^2}} = \arcsin\frac{x - a}{a}$$

265. $$\int \frac{x\,dx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a\arcsin\frac{x - a}{a}$$

266. $$\int \sqrt{2ax - x^2}\,dx = \frac{x - a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\arcsin\frac{x - a}{a}$$

267. $$\int \frac{dx}{(ax^2 + b)\sqrt{fx^2 + g}} = \frac{1}{\sqrt{b}\sqrt{ag - bf}}\arctan\frac{x\sqrt{ag - bf}}{\sqrt{b}\sqrt{fx^2 + g}}$$

$(ag - bf > 0)$

$$= \frac{1}{2\sqrt{b}\sqrt{bf - ag}}\log_e\frac{\sqrt{b}\sqrt{fx^2 + g} + x\sqrt{bf - ag}}{\sqrt{b}\sqrt{fx^2 + g} - x\sqrt{bf - ag}} \quad (ag - bf < 0)$$

268. $$\int \sqrt[n]{ax + b}\,dx = \frac{n(ax + b)}{(n + 1)a}\sqrt[n]{ax + b}$$

269. $$\int \frac{dx}{\sqrt[n]{ax + b}}\,dx = \frac{n(ax + b)}{(n - 1)a}\frac{1}{\sqrt[n]{ax + b}}$$

270. $$\int \frac{dx}{x\sqrt{x^n + a^2}} = -\frac{2}{na}\log_e\frac{a + \sqrt{x^n + a^2}}{\sqrt{x^n}}$$

271. $$\int \frac{dx}{x\sqrt{x^n - a^2}} = \frac{2}{na}\arccos\frac{a}{\sqrt{x^n}}$$

272. $$\int \frac{\sqrt{x}\,dx}{\sqrt{a^3 - x^3}} = \frac{2}{3}\arcsin\sqrt{\left(\frac{x}{a}\right)^3}$$

(q) *Recursion formulas* (m, n, p ***are integers***)

273. $$\int x^m(ax^n + b)^p\,dx$$

$$= \frac{1}{m + np + 1}\left[x^{m+1}(ax^n + b)^p + npb\int x^m(ax^n + b)^{p-1}\,dx\right]$$

$$= \frac{1}{bn(p+1)}\Big[-x^{m+1}(ax^n+b)^{p+1} + (m+n+np+1)\int x^m(ax^n+b)^{p+1}\,dx\Big]$$

$$= \frac{1}{(m+1)b}\Big[x^{m+1}(ax^n+b)^{p+1} - a(m+n+np+1)\int x^{m+n}(ax^n+b)^p\,dx\Big]$$

$$= \frac{1}{a(m+np+1)}\Big[x^{m-n+1}(ax^n+b)^{p+1} - (m-n+1)b\int x^{m-n}(ax^n+b)^p\,dx.\Big]$$

(r) *Integrals containing the sine function* $(a \neq 0)$

274. $\displaystyle\int \sin ax\,dx = -\frac{1}{a}\cos ax$

275. $\displaystyle\int \sin^2 ax\,dx = \frac{1}{2}x - \frac{1}{4a}\sin 2ax$

276. $\displaystyle\int \sin^3 ax\,dx = -\frac{1}{a}\cos ax + \frac{1}{3a}\cos^3 ax$

277. $\displaystyle\int \sin^4 ax\,dx = \frac{3}{8}x - \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax$

278. $\displaystyle\int \sin^n ax\,dx = -\frac{\sin^{n-1} ax\cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\,dx \quad (n>0)$

279. $\displaystyle\int x\sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$

280. $\displaystyle\int x^2\sin ax\,dx = \frac{2x}{a^2}\sin ax - \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\cos ax$

281. $\displaystyle\int x^3\sin ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax - \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\cos ax$

282. $\displaystyle\int x^n\sin ax\,dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx$

283. $\displaystyle\int \frac{\sin ax}{x}\,dx = ax - \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} - \frac{(ax)^7}{7\cdot 7!} + \cdots$

284. $$\int \frac{\sin ax}{x^2}\,dx = -\frac{\sin ax}{x} + a\int \frac{\cos ax\,dx}{x}$$

285. $$\int \frac{\sin ax}{x^n}\,dx = -\frac{1}{n-1}\frac{\sin ax}{x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}\,dx$$

286. $$\int \frac{dx}{\sin ax} = \int \operatorname{cosec} ax\,dx = \frac{1}{a}\log_e \tan\frac{ax}{2}$$
$$= \frac{1}{a}\log_e (\operatorname{cosec} ax - \cot ax)$$

287. $$\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$$

288. $$\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a\sin^2 ax} + \frac{1}{2a}\log_e \tan\frac{ax}{2}$$

289. $$\int \frac{dx}{\sin^n ax} = -\frac{1}{a(n-1)}\frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax} \qquad (n > 1)$$

290. $$\int \frac{x\,dx}{\sin ax} = \frac{1}{a^2}\left[ax + \frac{(ax)^3}{3\cdot 3!} + \frac{7(ax)^5}{3\cdot 5\cdot 5!} + \frac{31(ax)^7}{3\cdot 7\cdot 7!} + \frac{127(ax)^9}{3\cdot 5\cdot 9!} + \cdots\right]$$

291. $$\int \frac{x\,dx}{\sin^2 ax} = -\frac{x}{a}\cot ax + \frac{1}{a^2}\log_e \sin ax$$

292. $$\int \frac{x\,dx}{\sin^n ax} = -\frac{x\cos ax}{(n-1)a\sin^{n-1} ax} - \frac{1}{(n-1)(n-2)a^2\sin^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x\,dx}{\sin^{n-2} ax} \qquad (n > 2)$$

293. $$\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

294. $$\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

295. $$\int \frac{x\,dx}{1+\sin ax} = -\frac{x}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\log_e \cos\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

296. $$\int \frac{x\,dx}{1-\sin ax} = \frac{x}{a}\cot\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\log_e \sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

297. $$\int \frac{\sin ax\,dx}{1 \pm \sin ax} = \pm x + \frac{1}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right)$$

298. $$\int \frac{dx}{\sin ax\,(1 \pm \sin ax)} = \frac{1}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right) + \frac{1}{a}\log_e \tan\frac{ax}{2}$$

299. $$\int \frac{dx}{(1 + \sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

300. $$\int \frac{dx}{(1 - \sin ax)^2} = \frac{1}{2a}\cot\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a}\cot^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

301. $$\int \frac{\sin ax\,dx}{(1 + \sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

302. $$\int \frac{\sin ax\,dx}{(1 - \sin ax)^2} = -\frac{1}{2a}\cot\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a}\cot^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

303. $$\int \frac{dx}{1 + \sin^2 ax} = \frac{1}{2\sqrt{2}a}\arcsin\left(\frac{3\sin^2 ax - 1}{\sin^2 ax + 1}\right)$$

304. $$\int \frac{dx}{1 - \sin^2 ax} = \int \frac{dx}{\cos^2 ax} = \frac{1}{a}\tan ax$$

305. $$\int \sin ax \sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}$$

($|a| \neq |b|$; for $|a| = |b|$ see 275)

306. $$\int \frac{dx}{b + c\sin ax} = \frac{2}{a\sqrt{b^2 - c^2}}\arctan\frac{b\tan(ax/2) + c}{\sqrt{b^2 - c^2}} \qquad (b^2 > c^2)$$

$$= \frac{1}{a\sqrt{c^2 - b^2}}\log_e \frac{b\tan(ax/2) + c - \sqrt{c^2 - b^2}}{b\tan(ax/2) + c + \sqrt{c^2 - b^2}}$$

$(b^2 < c^2)$

307. $$\int \frac{\sin ax\,dx}{b + c\sin ax} = \frac{x}{c} - \frac{b}{c}\int \frac{dx}{b + c\sin ax}$$

308. $$\int \frac{dx}{\sin ax\,(b + c\sin ax)} = \frac{1}{ab}\log_e \tan\frac{ax}{2} - \frac{c}{b}\int \frac{dx}{b + c\sin ax}$$

309. $$\int \frac{dx}{(b + c\sin ax)^2} = \frac{c\cos ax}{a(b^2 - c^2)(b + c\sin ax)} + \frac{b}{b^2 - c^2}\int \frac{dx}{b + c\sin ax}$$

310. $$\int \frac{\sin ax\,dx}{(b+c\sin ax)^2} = \frac{b\cos ax}{a(c^2-b^2)(b+c\sin ax)} + \frac{c}{c^2-b^2}\int \frac{dx}{b+c\sin ax}$$

311. $$\int \frac{dx}{b^2+c^2\sin^2 ax} = \frac{1}{ab\sqrt{b^2+c^2}} \arctan \frac{\sqrt{b^2+c^2}\tan ax}{b} \qquad (b>0)$$

312. $$\int \frac{dx}{b^2-c^2\sin^2 ax} = \frac{1}{ab\sqrt{b^2-c^2}} \arctan \frac{\sqrt{b^2-c^2}\tan ax}{b} \qquad (b^2>c^2,\ b>0)$$

$$= \frac{1}{2ab\sqrt{c^2-b^2}} \log_e \frac{\sqrt{c^2-b^2}\tan ax + b}{\sqrt{c^2-b^2}\tan ax - b} \qquad (c^2>b^2,\ b>0)$$

(s) *Integrals containing the cosine function* $(a \neq 0)$

313. $$\int \cos ax\,dx = \frac{1}{a}\sin ax$$

314. $$\int \cos^2 ax\,dx = \frac{1}{2}x + \frac{1}{4a}\sin 2ax$$

315. $$\int \cos^3 ax\,dx = \frac{1}{a}\sin ax - \frac{1}{3a}\sin^3 ax$$

316. $$\int \cos^4 ax\,dx = \frac{3}{8}x + \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax$$

317. $$\int \cos^n ax\,dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx$$

318. $$\int x\cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$$

319. $$\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$$

320. $$\int x^3\cos ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\sin ax$$

321. $$\int x^n\cos ax\,dx = \frac{x^n\sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\,dx \qquad (n>0)$$

322. $$\int \frac{\cos ax}{x}\,dx = \log_e (ax) - \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^4}{4 \cdot 4!} - \frac{(ax)^6}{6 \cdot 6!} + \cdots$$

323. $$\int \frac{\cos ax}{x^2}\,dx = -\frac{\cos ax}{x} - a \int \frac{\sin ax\,dx}{x}$$

324. $$\int \frac{\cos ax}{x^n}\,dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1} \int \frac{\sin ax\,dx}{x^{n-1}} \qquad (n \neq 1)$$

325. $$\int \frac{dx}{\cos ax} = \frac{1}{a} \log_e \tan \left(\frac{ax}{2} + \frac{\pi}{4} \right) = \frac{1}{a} \log_e (\sec ax + \tan ax)$$

326. $$\int \frac{dx}{\cos^2 ax} = \frac{1}{a} \tan ax$$

327. $$\int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a \cos^2 ax} + \frac{1}{2a} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right)$$

328. $$\int \frac{dx}{\cos^n ax} = \frac{1}{a(n-1)} \frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} ax} \qquad (n > 1)$$

329. $$\int \frac{x\,dx}{\cos ax} = \frac{1}{a^2} \left[\frac{(ax)^2}{2} + \frac{(ax)^4}{4 \cdot 2!} + \frac{5(ax)^6}{6 \cdot 4!} + \frac{61(ax)^8}{8 \cdot 6!} + \frac{1{,}385(ax)^{10}}{10 \cdot 8!} + \cdots \right]$$

330. $$\int \frac{x\,dx}{\cos^2 ax} = \frac{x}{a} \tan ax + \frac{1}{a^2} \log_e \cos ax$$

331. $$\int \frac{x\,dx}{\cos^n ax} = \frac{x \sin ax}{(n-1)a \cos^{n-1} ax} - \frac{1}{(n-1)(n-2)a^2 \cos^{n-2} ax} + \frac{n-2}{n-1} \int \frac{x\,dx}{\cos^{n-2} ax} \qquad (n > 2)$$

332. $$\int \frac{dx}{1 + \cos ax} = \frac{1}{a} \tan \frac{ax}{2}$$

333. $$\int \frac{dx}{1 - \cos ax} = -\frac{1}{a} \cot \frac{ax}{2}$$

334. $$\int \frac{x\,dx}{1 + \cos ax} = \frac{x}{a} \tan \frac{ax}{2} + \frac{2}{a^2} \log_e \cos \frac{ax}{2}$$

335. $$\int \frac{x\,dx}{1 - \cos ax} = -\frac{x}{a} \cot \frac{ax}{2} + \frac{2}{a^2} \log_e \sin \frac{ax}{2}$$

336. $$\int \frac{\cos ax\,dx}{1+\cos ax} = x - \frac{1}{a}\tan\frac{ax}{2}$$

337. $$\int \frac{\cos ax\,dx}{1-\cos ax} = -x - \frac{1}{a}\cot\frac{ax}{2}$$

338. $$\int \frac{dx}{\cos ax\,(1+\cos ax)} = \frac{1}{a}\log_e \tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) - \frac{1}{a}\tan\frac{ax}{2}$$

339. $$\int \frac{dx}{\cos ax\,(1-\cos ax)} = \frac{1}{a}\log_e \tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) - \frac{1}{a}\cot\frac{ax}{2}$$

340. $$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^3\frac{ax}{2}$$

341. $$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$$

342. $$\int \frac{\cos ax\,dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} - \frac{1}{6a}\tan^3\frac{ax}{2}$$

343. $$\int \frac{\cos ax\,dx}{(1-\cos ax)^2} = \frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$$

344. $$\int \frac{dx}{1+\cos^2 ax} = \frac{1}{2\sqrt{2}a}\arcsin\left(\frac{1-3\cos^2 ax}{1+\cos^2 ax}\right)$$

345. $$\int \frac{dx}{1-\cos^2 ax} = \int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$$

346. $$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}$$

($|a| \neq |b|$; for $|a| = |b|$, see 314)

347. $$\int \frac{dx}{b+c\cos ax} = \frac{2}{a\sqrt{b^2-c^2}}\arctan\frac{(b-c)\tan(ax/2)}{\sqrt{b^2-c^2}} \qquad (b^2 > c^2)$$

$$= \frac{1}{a\sqrt{c^2-b^2}}\log_e\frac{(c-b)\tan(ax/2)+\sqrt{c^2-b^2}}{(c-b)\tan(ax/2)-\sqrt{c^2-b^2}}$$

$(b^2 < c^2)$

348. $$\int \frac{\cos ax\,dx}{b+c\cos ax} = \frac{x}{c} - \frac{b}{c}\int \frac{dx}{b+c\cos ax}$$

349. $$\int \frac{dx}{\cos ax\,(b + c\cos ax)} = \frac{1}{ab}\log_e \tan\left(\frac{ax}{2} + \frac{\pi}{4}\right) - \frac{c}{b}\int \frac{dx}{b + c\cos ax}$$

350. $$\int \frac{dx}{(b + c\cos ax)^2} = \frac{c\sin ax}{a(c^2 - b^2)(b + c\cos ax)} - \frac{b}{c^2 - b^2}\int \frac{dx}{b + c\cos ax}$$

351. $$\int \frac{\cos ax\,dx}{(b + c\cos ax)^2} = \frac{b\sin ax}{a(b^2 - c^2)(b + c\cos ax)} - \frac{c}{b^2 - c^2}\int \frac{dx}{b + c\cos ax}$$

352. $$\int \frac{dx}{b^2 + c^2\cos^2 ax} = \frac{1}{ab\sqrt{b^2 + c^2}}\arctan\frac{b\tan ax}{\sqrt{b^2 + c^2}} \qquad (b > 0)$$

353. $$\int \frac{dx}{b^2 - c^2\cos^2 ax} = \frac{1}{ab\sqrt{b^2 - c^2}}\arctan\frac{b\tan ax}{\sqrt{b^2 - c^2}} \qquad (b^2 > c^2, b > 0)$$

$$= \frac{1}{2ab\sqrt{c^2 - b^2}}\log_e \frac{b\tan ax - \sqrt{c^2 - b^2}}{b\tan ax + \sqrt{c^2 - b^2}} \qquad (c^2 > b^2, b > 0)$$

(t) *Integrals containing both sine and cosine* $(a \neq 0)$

354. $$\int \sin ax\cos ax\,dx = \frac{1}{2a}\sin^2 ax$$

355. $$\int \sin^2 ax\cos^2 ax\,dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

356. $$\int \sin^n ax\cos ax\,dx = \frac{1}{a(n+1)}\sin^{n+1} ax \qquad (n \neq -1)$$

357. $$\int \sin ax\cos^n ax\,dx = -\frac{1}{a(n+1)}\cos^{n+1} ax \qquad (n \neq -1)$$

358. $$\int \sin^n ax\cos^m ax\,dx = -\frac{\sin^{n-1} ax\cos^{m+1} ax}{a(n+m)} + \frac{n-1}{n+m}\int \sin^{n-2} ax\cos^m ax\,dx \qquad (m, n > 0)$$

$$= \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(n+m)} + \frac{m-1}{n+m} \int \sin^n ax \cos^{m-2} ax \, dx$$

$(m, n > 0)$

359. $$\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a} \log_e \tan ax$$

360. $$\int \frac{dx}{\sin^2 ax \cos ax} = \frac{1}{a}\left[\log_e \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) - \frac{1}{\sin ax}\right]$$

361. $$\int \frac{dx}{\sin ax \cos^2 ax} = \frac{1}{a}\left(\log_e \tan \frac{ax}{2} + \frac{1}{\cos ax}\right)$$

362. $$\int \frac{dx}{\sin^3 ax \cos ax} = \frac{1}{a}\left(\log_e \tan ax - \frac{1}{2 \sin^2 ax}\right)$$

363. $$\int \frac{dx}{\sin ax \cos^3 ax} = \frac{1}{a}\left(\log_e \tan ax + \frac{1}{2 \cos^2 ax}\right)$$

364. $$\int \frac{dx}{\sin^2 ax \cos^2 ax} = -\frac{2}{a} \cot 2ax$$

365. $$\int \frac{dx}{\sin^2 ax \cos^3 ax} = \frac{1}{a}\left[\frac{\sin ax}{2 \cos^2 ax} - \frac{1}{\sin ax} + \frac{3}{2} \log_e \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right]$$

366. $$\int \frac{dx}{\sin^3 ax \cos^2 ax} = \frac{1}{a}\left(\frac{1}{\cos ax} - \frac{\cos ax}{2 \sin^2 ax} + \frac{3}{2} \log_e \tan \frac{ax}{2}\right)$$

367. $$\int \frac{dx}{\sin ax \cos^n ax} = \frac{1}{a(n-1) \cos^{n-1} ax} + \int \frac{dx}{\sin ax \cos^{n-2} ax}$$

$(n \neq 1)$; (see 361, 363)

368. $$\int \frac{dx}{\sin^n ax \cos ax} = -\frac{1}{a(n-1) \sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax \cos ax}$$

$(n \neq 1)$; (see 360, 362)

369. $$\int \frac{dx}{\sin^n ax \cos^m ax}$$

$$= -\frac{1}{a(n-1)} \cdot \frac{1}{\sin^{n-1} ax \cos^{m-1} ax} + \frac{n+m-2}{n-1} \int \frac{dx}{\sin^{n-2} ax \cos^m ax}$$

$(m > 0, n > 1)$

$$= \frac{1}{a(m-1)} \cdot \frac{1}{\sin^{n-1} ax \cos^{m-1} ax} + \frac{n+m-2}{m-1} \int \frac{dx}{\sin^n ax \cos^{m-2} ax}$$

$(n > 0, m > 1)$

370. $$\int \frac{\sin ax\,dx}{\cos^2 ax} = \frac{1}{a\cos ax}$$

371. $$\int \frac{\sin ax\,dx}{\cos^3 ax} = \frac{1}{2a\cos^2 ax} = \frac{1}{2a}\tan^2 ax + C_1$$

372. $$\int \frac{\sin ax\,dx}{\cos^n ax} = \frac{1}{a(n-1)\cos^{n-1} ax}$$

373. $$\int \frac{\sin^2 ax\,dx}{\cos ax} = -\frac{1}{a}\sin ax + \frac{1}{a}\log_e \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

374. $$\int \frac{\sin^2 ax\,dx}{\cos^3 ax} = \frac{1}{a}\left[\frac{\sin ax}{2\cos^2 ax} - \frac{1}{2}\log_e \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right]$$

375. $$\int \frac{\sin^2 ax\,dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{n-1}\int \frac{dx}{\cos^{n-2} ax}$$

$(n \neq 1)$ (see 325, 326, 328)

376. $$\int \frac{\sin^3 ax\,dx}{\cos ax} = -\frac{1}{a}\left(\frac{\sin^2 ax}{2} + \log_e \cos ax\right)$$

377. $$\int \frac{\sin^3 ax\,dx}{\cos^2 ax} = \frac{1}{a}\left(\cos ax + \frac{1}{\cos ax}\right)$$

378. $$\int \frac{\sin^3 ax\,dx}{\cos^n ax} = \frac{1}{a}\left[\frac{1}{(n-1)\cos^{n-1} ax} - \frac{1}{(n-3)\cos^{n-3} ax}\right]$$

$(n \neq 1, n \neq 3)$

379. $$\int \frac{\sin^n ax}{\cos ax}\,dx = -\frac{\sin^{n-1} ax}{a(n-1)} + \int \frac{\sin^{n-2} ax\,dx}{\cos ax} \qquad (n \neq 1)$$

380. $$\int \frac{\sin^n ax}{\cos^m ax}\,dx = \frac{\sin^{n+1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-m+2}{m-1}\int \frac{\sin^n ax}{\cos^{m-2} ax}\,dx$$

$(m \neq 1)$

$$= -\frac{\sin^{n-1} ax}{a(n-m)\cos^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\sin^{n-2} ax\,dx}{\cos^m ax}$$

$(m \neq n)$

$$= \frac{\sin^{n-1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\sin^{n-1} ax\,dx}{\cos^{m-2} ax}$$

$(m \neq 1)$

381. $$\int \frac{\cos ax\,dx}{\sin^2 ax} = -\frac{1}{a \sin ax}$$

382. $$\int \frac{\cos ax\,dx}{\sin^3 ax} = -\frac{1}{2a \sin^2 ax}$$

383. $$\int \frac{\cos ax\,dx}{\sin^n ax} = -\frac{1}{a(n-1)\sin^{n-1} ax} \qquad (n \neq 1)$$

384. $$\int \frac{\cos^2 ax\,dx}{\sin ax} = \frac{1}{a}\left(\cos ax + \log_e \tan \frac{ax}{2}\right)$$

385. $$\int \frac{\cos^2 ax\,dx}{\sin^3 ax} = -\frac{1}{2a}\left(\frac{\cos ax}{\sin^2 ax} - \log_e \tan \frac{ax}{2}\right)$$

386. $$\int \frac{\cos^2 ax\,dx}{\sin^n ax} = -\frac{1}{n-1}\left(\frac{\cos ax}{a \sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax}\right)$$

$(n \neq 1)$; (see 289)

387. $$\int \frac{\cos^3 ax\,dx}{\sin ax} = \frac{1}{a}\left(\frac{\cos^2 ax}{2} + \log_e \sin ax\right)$$

388. $$\int \frac{\cos^3 ax\,dx}{\sin^2 ax} = -\frac{1}{a}\left(\sin ax + \frac{1}{\sin ax}\right)$$

389. $$\int \frac{\cos^3 ax\,dx}{\sin^n ax} = \frac{1}{a}\left[\frac{1}{(n-3)\sin^{n-3} ax} - \frac{1}{(n-1)\sin^{n-1} ax}\right]$$

$(n \neq 1, n \neq 3)$

390. $$\int \frac{\cos^n ax}{\sin ax}\,dx = \frac{\cos^{n-1} ax}{a(n-1)} + \int \frac{\cos^{n-2} ax\,dx}{\sin ax} \qquad (n \neq 1)$$

391. $$\int \frac{\cos^n ax\,dx}{\sin^m ax} = -\frac{\cos^{n+1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-m+2}{m-1}\int \frac{\cos^n ax\,dx}{\sin^{m-2} ax}$$

$(m \neq 1)$

$$= \frac{\cos^{n-1} ax}{a(n-m)\sin^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\cos^{n-2} ax\,dx}{\sin^m ax}$$

$(m \neq n)$

$$= -\frac{\cos^{n-1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\cos^{n-2} ax\,dx}{\sin^{m-2} ax}$$

$(m \neq 1)$

392. $$\int \frac{dx}{\sin ax\,(1 \pm \cos ax)} = \pm\frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a}\log_e \tan\frac{ax}{2}$$

393. $$\int \frac{dx}{\cos ax\,(1 \pm \sin ax)} = \mp\frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a}\log_e \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

394. $$\int \frac{\sin ax\,dx}{\cos ax\,(1 \pm \cos ax)} = \frac{1}{a}\log_e \frac{1 \pm \cos ax}{\cos ax}$$

395. $$\int \frac{\cos ax\,dx}{\sin ax\,(1 \pm \sin ax)} = -\frac{1}{a}\log_e \frac{1 \pm \sin ax}{\sin ax}$$

396. $$\int \frac{\sin ax\,dx}{\cos ax\,(1 \pm \sin ax)} = \frac{1}{2a(1 \pm \sin ax)} \pm \frac{1}{2a}\log_e \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

397. $$\int \frac{\cos ax\,dx}{\sin ax\,(1 \pm \cos ax)} = -\frac{1}{2a(1 \pm \cos ax)} \pm \frac{1}{2a}\log_e \tan\frac{ax}{2}$$

398. $$\int \frac{\sin ax\,dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \frac{1}{2a}\log_e (\sin ax \pm \cos ax)$$

399. $$\int \frac{\cos ax\,dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a}\log_e (\sin ax \pm \cos ax)$$

400. $$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}}\log_e \tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)$$

401. $$\int \frac{dx}{1 + \cos ax \pm \sin ax} = \pm\frac{1}{a}\log_e \left(1 \pm \tan\frac{ax}{2}\right)$$

402. $$\int \frac{dx}{b\sin ax + c\cos ax} = \frac{1}{a\sqrt{b^2 + c^2}}\log_e \tan\frac{ax + \theta}{2}$$

$$\sin\theta = \frac{c}{\sqrt{b^2 + c^2}} \qquad \tan\theta = \frac{c}{b}$$

403. $$\int \frac{\sin ax\,dx}{b + c\cos ax} = -\frac{1}{ac}\log_e (b + c\cos ax)$$

404. $$\int \frac{\cos ax\,dx}{b + c\sin ax} = \frac{1}{ac}\log_e (b + c\sin ax)$$

405. $$\int \frac{\sin^2 ax\,dx}{b + c\cos^2 ax} = \frac{1}{ac}\sqrt{\frac{b + c}{b}}\arctan\left(\sqrt{\frac{b}{b + c}}\tan ax\right) - \frac{x}{c}$$

406. $$\int \frac{\sin ax \cos ax\, dx}{b\cos^2 ax + c\sin^2 ax} = \frac{1}{2a(c-b)} \log_e (b\cos^2 ax + c\sin^2 ax)$$

$(c \neq b)$

407. $$\int \frac{dx}{b^2\cos^2 ax + c^2\sin^2 ax} = \frac{1}{abc} \arctan\left(\frac{c}{b}\tan ax\right)$$

408. $$\int \frac{dx}{b^2\cos^2 ax - c^2\sin^2 ax} = \frac{1}{2abc} \log_e \frac{c\tan ax + b}{c\tan ax - b}$$

409. $$\int \sin ax \cos bx\, dx = -\frac{\cos (a+b)x}{2(a+b)} - \frac{\cos (a-b)x}{2(a-b)}$$

$(a^2 \neq b^2$; for $a = b$ see 354)

410. $$\int \frac{dx}{b + c\cos ax + d\sin ax}$$

$$= \begin{cases} \dfrac{-1}{a\sqrt{b^2 - c^2 - d^2}} \arcsin \dfrac{c^2 + d^2 + b(c\cos ax + d\sin ax)}{\sqrt{c^2 + d^2}(b + c\cos ax + d\sin ax)} \\ \qquad\qquad (b^2 > c^2 + d^2,\ |ax| < \pi) \\ \dfrac{1}{a\sqrt{c^2 + d^2 - b^2}} \\ \quad \log_e \dfrac{c^2 + d^2 + b(c\cos ax + d\sin ax) + \sqrt{c^2 + d^2 - b^2}(c\sin ax - d\cos ax)}{\sqrt{c^2 + d^2}(b + c\cos ax + d\sin ax)} \\ \qquad\qquad (b^2 < c^2 + d^2,\ |ax| < \pi) \\ \dfrac{1}{ab}\left[\dfrac{b - (c+d)\cos ax + (c-d)\sin ax}{b + (c-d)\cos ax + (c+d)\sin ax}\right] \qquad (b^2 = c^2 + d^2) \end{cases}$$

(u) Integrals containing tangent and cotangent functions $(a \neq 0)$

411. $$\int \tan ax\, dx = -\frac{1}{a}\log_e \cos ax$$

412. $$\int \tan^2 ax\, dx = \frac{1}{a}\tan ax - x$$

413. $$\int \tan^3 ax\, dx = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\log_e \cos ax$$

414. $$\int \tan^n ax\, dx = \frac{1}{a(n-1)}\tan^{n-1} ax - \int \tan^{n-2} ax\, dx \qquad (n > 1)$$

415. $$\int \frac{dx}{b + c\tan ax} = \int \frac{\cot ax\,dx}{b\cot ax + c}$$
$$= \frac{1}{b^2 + c^2}\left[bx + \frac{c}{a}\log_e (b\cos ax + c\sin ax)\right]$$

416. $$\int \frac{dx}{\sqrt{b + c\tan^2 ax}} = \frac{1}{a\sqrt{b - c}}\arcsin\left(\sqrt{\frac{b - c}{b}}\sin ax\right)$$
$(b > 0, b^2 > c^2)$

417. $$\int \frac{\tan^n ax}{\cos^2 ax}dx = \frac{\tan^{n+1} ax}{a(n + 1)} \qquad (n \neq -1)$$

418. $$\int \cot ax\,dx = \frac{1}{a}\log_e \sin ax$$

419. $$\int \cot^2 ax\,dx = \int \frac{dx}{\tan^2 ax} = -\frac{1}{a}\cot ax - x$$

420. $$\int \cot^3 ax\,dx = -\frac{1}{2a}\cot^2 ax - \frac{1}{a}\log_e \sin ax$$

421. $$\int \cot^n ax\,dx = \int \frac{dx}{\tan^n ax} = -\frac{1}{a(n - 1)}\cot^{n-1} ax$$
$$- \int \cot^{n-2} ax\,dx \qquad (n > 1)$$

422. $$\int \frac{dx}{b + c\cot ax} = \int \frac{\tan ax\,dx}{b\tan ax + c}$$
$$= \frac{1}{b^2 + c^2}\left[bx - \frac{c}{a}\log_e (c\cos ax + b\sin ax)\right]$$

423. $$\int \frac{\cot^n ax}{\sin^2 ax}dx = \frac{-\cot^{n+1} ax}{a(n + 1)} \qquad (n \neq -1)$$

(v) *Integrals containing hyperbolic functions* $(a \neq 0)$

424. $$\int \sinh x\,dx = \cosh x$$

425. $$\int \sinh^2 x\,dx = \frac{\sinh 2x}{4} - \frac{x}{2}$$

426. $$\int \frac{dx}{\sinh x} = \log_e \tanh\left(\frac{x}{2}\right)$$

427. $\int \frac{dx}{\sinh^2 x} = -\coth x$

428. $\int \cosh x\, dx = \sinh x$

429. $\int \cosh^2 x\, dx = \frac{\sinh 2x}{4} + \frac{x}{2}$

430. $\int \frac{dx}{\cosh x} = 2 \arctan e^x = \arctan (\sinh x)$

431. $\int \frac{dx}{\cosh^2 x} = \tanh x$

432. $\int \frac{\sinh x}{\cosh^2 x}\, dx = -\frac{1}{\cosh x}$

433. $\int \frac{\cosh x}{\sinh^2 x}\, dx = -\frac{1}{\sinh x}$

434. $\int x \sinh x\, dx = x \cosh x - \sinh x$

435. $\int x \cosh x\, dx = x \sinh x - \cosh x$

436. $\int \tanh x\, dx = \log_e \cosh x$

437. $\int \tanh^2 x\, dx = x - \tanh x$

438. $\int \coth x\, dx = \log_e \sinh x$

439. $\int \coth^2 x\, dx = x - \coth x$

440. $\int \sinh^n ax\, dx$

$$= \begin{cases} \frac{1}{an} \sinh^{n-1} ax \cosh ax - \frac{n-1}{n} \int \sinh^{n-2} ax\, dx & (n > 0) \\ \frac{1}{a(n+1)} \sinh^{n+1} ax \cosh ax - \frac{n+2}{n+1} \int \sinh^{n+2} ax\, dx & (n < -1) \end{cases}$$

441. $$\int \cosh^n ax\, dx$$
$$= \begin{cases} \dfrac{1}{an}\sinh ax \cosh^{n-1} ax + \dfrac{n-1}{n}\displaystyle\int \cosh^{n-2} ax\, dx & (n > 0) \\ -\dfrac{1}{a(n+1)}\sinh ax \cosh^{n+1} ax + \dfrac{n+2}{n+1}\displaystyle\int \cosh^{n+2} ax\, dx & (n < -1) \end{cases}$$

442. $$\int \sinh ax \sinh bx\, dx = \frac{\sinh (a+b)x}{2(a+b)} - \frac{\sinh (a-b)x}{2(a-b)}$$

443. $$\int \cosh ax \cosh bx\, dx = \frac{\sinh (a+b)x}{2(a+b)} + \frac{\sinh (a-b)x}{2(a-b)}$$

444. $$\int \sinh ax \cosh bx\, dx = \frac{\cosh (a+b)x}{2(a+b)} + \frac{\cosh (a-b)x}{2(a-b)} \qquad (a^2 \neq b^2)$$

445. $$\int \sinh ax \sin ax\, dx = \frac{1}{2a}(\cosh ax \sin ax - \sinh ax \cos ax)$$

446. $$\int \cosh ax \cos ax\, dx = \frac{1}{2a}(\sinh ax \cos ax + \cosh ax \sin ax)$$

447. $$\int \sinh ax \cos ax\, dx = \frac{1}{2a}(\cosh ax \cos ax + \sinh ax \sin ax)$$

448. $$\int \cosh ax \sin ax\, dx = \frac{1}{2a}(\sinh ax \sin ax - \cosh ax \cos ax)$$

(w) *Integrals containing exponential functions*

449. $$\int e^{ax}\, dx = \frac{1}{a}e^{ax}$$

450. $$\int xe^{ax}\, dx = \frac{e^{ax}}{a^2}(ax - 1)$$

451. $$\int x^2 e^{ax}\, dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$$

452. $$\int x^n e^{ax}\, dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx \qquad (n > 0)$$

453. $$\int \frac{e^{ax}}{x}\, dx = \log_e x + \frac{ax}{1 \cdot 1!} + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^3}{3 \cdot 3!} + \cdots$$

454. $$\int \frac{e^{ax}}{x^n}\,dx = \frac{1}{n-1}\left(-\frac{e^{ax}}{x^{n-1}} + a\int \frac{e^{ax}}{x^{n-1}}\,dx\right) \qquad (n > 1)$$

455. $$\int \frac{dx}{1+e^{ax}} = \frac{1}{a}\log_e \frac{e^{ax}}{1+e^{ax}}$$

456. $$\int \frac{dx}{b+ce^{ax}} = \frac{x}{b} - \frac{1}{ab}\log_e (b+ce^{ax})$$

457. $$\int \frac{e^{ax}\,dx}{b+ce^{ax}} = \frac{1}{ac}\log_e (b+ce^{ax})$$

458. $$\int \frac{dx}{be^{ax}+ce^{-ax}} = \frac{1}{a\sqrt{bc}}\arctan\left(e^{ax}\sqrt{\frac{b}{c}}\right) \qquad (bc > 0)$$

$$= \frac{1}{2a\sqrt{-bc}}\log_e \frac{c+e^{ax}\sqrt{-bc}}{c-e^{ax}\sqrt{-bc}} \qquad (bc < 0)$$

459. $$\int \frac{xe^{ax}\,dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}$$

460. $$\int e^{ax}\log_e x\,dx = \frac{1}{a}e^{ax}\log_e x - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx$$

461. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

462. $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

463. $$\int xe^{ax}\sin bx\,dx = \frac{xe^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$- \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\sin bx - 2ab\cos bx]$$

464. $$\int xe^{ax}\cos bx\,dx = \frac{xe^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$- \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\cos bx + 2ab\sin bx]$$

465. $$\int e^{ax}\sin bx\sin cx\,dx = \frac{e^{ax}[(b-c)\sin(b-c)x + a\cos(b-c)x]}{2[a^2+(b-c)^2]}$$

$$- \frac{e^{ax}[(b+c)\sin(b+c)x + a\cos(b+c)x]}{2[a^2+(b+c)^2]}$$

466. $$\int e^{ax}\cos bx\cos cx\,dx = \frac{e^{ax}[(b-c)\sin(b-c)x + a\cos(b-c)x]}{2[a^2+(b-c)^2]} + \frac{e^{ax}[(b+c)\sin(b+c)x + a\cos(b+c)x]}{2[a^2+(b+c)^2]}$$

467. $$\int e^{ax}\sin bx\cos cx\,dx = \frac{e^{ax}[a\sin(b-c)x - (b-c)\cos(b-c)x]}{2[a^2+(b-c)^2]} + \frac{e^{ax}[a\sin(b+c)x - (b+c)\cos(b+c)x]}{2[a^2+(b+c)^2]}$$

468. $$\int e^{ax}\sin bx\sin(bx+c)\,dx = \frac{e^{ax}\cos c}{2a} - \frac{e^{ax}[a\cos(2bx+c) + 2b\sin(2bx+c)]}{2(a^2+4b^2)}$$

469. $$\int e^{ax}\cos bx\cos(bx+c)\,dx = \frac{e^{ax}\cos c}{2a} + \frac{e^{ax}[a\cos(2bx+c) + 2b\sin(2bx+c)]}{2(a^2+4b^2)}$$

470. $$\int e^{ax}\sin bx\cos(bx+c)\,dx = -\frac{e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx+c) - 2b\cos(2bx+c)]}{2(a^2+4b^2)}$$

471. $$\int e^{ax}\cos bx\sin(bx+c)\,dx = \frac{e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx+c) - 2b\cos(2bx+c)]}{2(a^2+4b^2)}$$

472. $$\int e^{ax}\sin^n bx\,dx = \frac{e^{ax}\sin^{n-1}bx\,(a\sin bx - nb\cos bx)}{a^2+n^2b^2} + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\sin^{n-2}bx\,dx$$

473. $$\int e^{ax}\cos^n bx\,dx = \frac{e^{ax}\cos^{n-1}bx\,(a\cos bx + nb\sin bx)}{a^2+n^2b^2} + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\cos^{n-2}bx\,dx$$

(x) *Integrals containing logarithmic functions* $(a \neq 0)$

474. $\int \log_e ax\, dx = x \log_e ax - x$

475. $\int (\log_e ax)^2\, dx = x(\log_e ax)^2 - 2x \log_e ax + 2x$

476. $\int (\log_e ax)^n\, dx = x(\log_e ax)^n - n \int (\log_e ax)^{n-1}\, dx \qquad (n \neq -1)$

477. $\int \frac{dx}{\log_e ax} = \frac{1}{a}\left[\log_e (\log_e ax) + \log_e ax + \frac{(\log_e ax)^2}{2 \cdot 2!} + \frac{(\log_e ax)^3}{3 \cdot 3!} + \cdots\right]$

478. $\int x \log_e ax\, dx = \frac{x^2}{2} \log_e ax - \frac{x^2}{4}$

479. $\int x^2 \log_e ax\, dx = \frac{x^3}{3} \log_e ax - \frac{x^3}{9}$

480. $\int x^n \log_e ax\, dx = x^{n+1}\left[\frac{\log_e ax}{n+1} - \frac{1}{(n+1)^2}\right] \qquad (n \neq -1)$

481. $\int x^n (\log_e ax)^m\, dx = \frac{x^{n+1}}{n+1} (\log_e ax)^m - \frac{m}{n+1} \int x^n (\log_e ax)^{m-1}\, dx \qquad (m, n \neq -1)$

482. $\int \frac{(\log_e ax)^n}{x}\, dx = \frac{(\log_e ax)^{n+1}}{n+1} \qquad (n \neq -1)$

483. $\int \frac{\log_e x}{x^n}\, dx = -\frac{\log_e x}{(n-1)x^{n-1}} - \frac{1}{(n-1)^2 x^{n-1}} \qquad (n \neq 1)$

484. $\int \frac{(\log_e x)^m}{x^n}\, dx = -\frac{(\log_e x)^m}{(n-1)x^{n-1}} + \frac{m}{n-1} \int \frac{(\log_e x)^{m-1}}{x^n}\, dx \qquad (n \neq 1)$

485. $\int \frac{x^n\, dx}{\log_e ax} = \frac{1}{a^{n+1}}\left[\log_e (\log_e ax) + (n+1) \log_e ax + \frac{(n+1)^2 (\log_e ax)^2}{2 \cdot 2!} + \frac{(n+1)^3 (\log_e ax)^3}{3 \cdot 3!} + \cdots\right]$

$= \frac{1}{a^{n+1}} \int \frac{e^y\, dy}{y} \qquad [y = (n+1) \log_e ax]$

486. $$\int \frac{x^n\,dx}{(\log_e ax)^m} = \frac{-x^{n+1}}{(m-1)(\log_e ax)^{m-1}} + \frac{n+1}{m-1}\int \frac{x^n\,dx}{(\log_e ax)^{m-1}}$$

$(m \neq 1)$

487. $$\int \frac{dx}{x \log_e ax} = \log_e (\log_e ax)$$

488. $$\int \frac{dx}{x(\log_e ax)^n} = -\frac{1}{(n-1)(\log_e ax)^{n-1}}$$

489. $$\int \sin(\log_e ax)\,dx = \frac{x}{2}[\sin(\log_e ax) - \cos(\log_e ax)]$$

490. $$\int \cos(\log_e ax)\,dx = \frac{x}{2}[\sin(\log_e ax) + \cos(\log_e ax)]$$

491. $$\int e^{ax} \log_e bx\,dx = \frac{1}{a} e^{ax} \log_e bx - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx$$

(y) *Integrals containing inverse trigonometric and hyperbolic functions* $(a > 0)$

492. $$\int \arcsin\frac{x}{a}\,dx = x \arcsin\frac{x}{a} + \sqrt{a^2 - x^2}$$

493. $$\int x \arcsin\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2 - x^2}$$

494. $$\int x^2 \arcsin\frac{x}{a}\,dx = \frac{x^3}{3}\arcsin\frac{x}{a} + \frac{1}{9}(x^2 + 2a^2)\sqrt{a^2 - x^2}$$

495. $$\int x^n \arcsin\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arcsin\frac{x}{a} - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{a^2 - x^2}}\,dx$$

$(n \neq -1)$

496. $$\int \frac{\arcsin\frac{x}{a}\,dx}{x} = \frac{x}{a} + \frac{1}{2\cdot 3\cdot 3}\frac{x^3}{a^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\frac{x^5}{a^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\frac{x^7}{a^7} + \cdots$$

$(x^2 < a^2)$

497. $$\int \frac{\arcsin\frac{x}{a}\,dx}{x^2} = -\frac{1}{x}\arcsin\frac{x}{a} - \frac{1}{a}\log_e \frac{a + \sqrt{a^2 - x^2}}{x}$$

498. $$\int \left(\arcsin\frac{x}{a}\right)^2 dx = x\left(\arcsin\frac{x}{a}\right)^2 + 2\left(\sqrt{a^2-x^2}\arcsin\frac{x}{a} - x\right)$$

499. $$\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2}$$

500. $$\int x\arccos\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2-x^2}$$

501. $$\int x^2\arccos\frac{x}{a}\,dx = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}$$

502. $$\int x^n\arccos\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arccos\frac{x}{a} + \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{a^2-x^2}}\,dx$$

$(n \neq -1)$

503. $$\int \frac{\arccos\frac{x}{a}\,dx}{x} = \frac{\pi}{2}\log_e x - \frac{x}{a} - \frac{1}{2\cdot 3\cdot 3}\frac{x^3}{a^3} - \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\frac{x^5}{a^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\frac{x^7}{a^7} - \cdots$$

$(x^2 < a^2)$

504. $$\int \frac{\arccos\frac{x}{a}\,dx}{x^2} = -\frac{1}{x}\arccos\frac{x}{a} + \frac{1}{a}\log_e\frac{a+\sqrt{a^2-x^2}}{x}$$

505. $$\int \left(\arccos\frac{x}{a}\right)^2 dx = x\left(\arccos\frac{x}{a}\right)^2 - 2\left(\sqrt{a^2-x^2}\arccos\frac{x}{a} + x\right)$$

506. $$\int \arctan\frac{x}{a}\,dx = x\arctan\frac{x}{a} - \frac{a}{2}\log_e(a^2+x^2)$$

507. $$\int x\arctan\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\arctan\frac{x}{a} - \frac{ax}{2}$$

508. $$\int x^2\arctan\frac{x}{a}\,dx = \frac{x^3}{3}\arctan\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\log_e(a^2+x^2)$$

509. $$\int x^n\arctan\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arctan\frac{x}{a} - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{a^2+x^2}$$

$(n \neq -1)$

510. $$\int \frac{\arctan \frac{x}{a}\, dx}{x} = \frac{x}{a} - \frac{x^3}{3^2a^3} + \frac{x^5}{5^2a^5} - \frac{x^7}{7^2a^7} + \cdots \quad (|x| < |a|)$$

511. $$\int \frac{\arctan \frac{x}{a}\, dx}{x^2} = -\frac{1}{x} \arctan \frac{x}{a} - \frac{1}{2a} \log_e \frac{a^2 + x^2}{x^2}$$

512. $$\int \frac{\arctan \frac{x}{a}\, dx}{x^n} = -\frac{1}{(n-1)x^{n-1}} \arctan \frac{x}{a} + \frac{a}{n-1} \int \frac{dx}{x^{n-1}(a^2 + x^2)} \qquad (n \neq 1)$$

513. $$\int \operatorname{arccot} \frac{x}{a}\, dx = x \operatorname{arccot} \frac{x}{a} + \frac{a}{2} \log_e (a^2 + x^2)$$

514. $$\int x \operatorname{arccot} \frac{x}{a}\, dx = \frac{1}{2}(x^2 + a^2) \operatorname{arccot} \frac{x}{a} + \frac{ax}{2}$$

515. $$\int x^2 \operatorname{arccot} \frac{x}{a}\, dx = \frac{x^3}{3} \operatorname{arccot} \frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6} \log_e (a^2 + x^2)$$

516. $$\int x^n \operatorname{arccot} \frac{x}{a}\, dx = \frac{x^{n+1}}{n+1} \operatorname{arccot} \frac{x}{a} + \frac{a}{n+1} \int \frac{x^{n+1}\, dx}{a^2 + x^2} \qquad (n \neq -1)$$

517. $$\int \frac{\operatorname{arccot} \frac{x}{a}\, dx}{x} = \frac{\pi}{2} \log_e x - \frac{x}{a} + \frac{x^3}{3^2a^3} - \frac{x^5}{5^2a^5} + \frac{x^7}{7^2a^7} - \cdots$$

518. $$\int \frac{\operatorname{arccot} \frac{x}{a}\, dx}{x^2} = -\frac{1}{x} \operatorname{arccot} \frac{x}{a} + \frac{1}{2a} \log_e \frac{a^2 + x^2}{x^2}$$

519. $$\int \frac{\operatorname{arccot} \frac{x}{a}\, dx}{x^n} = -\frac{1}{(n-1)x^{n-1}} \operatorname{arccot} \frac{x}{a} - \frac{a}{n-1} \int \frac{dx}{x^{n-1}(a^2 + x^2)} \qquad (n \neq 1)$$

520. $$\int \sinh^{-1}\frac{x}{a}\,dx = x\sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2}$$

521. $$\int x\sinh^{-1}\frac{x}{a}\,dx = \frac{1}{2}\left(x^2+\frac{a^2}{2}\right)\sinh^{-1}\frac{x}{a} - \frac{x}{4}\sqrt{x^2+a^2}$$

522. $$\int \cosh^{-1}\frac{x}{a}\,dx = x\cosh^{-1}\frac{x}{a} \mp \sqrt{x^2-a^2}$$

$$\left(\text{upper sign for } \cosh^{-1}\frac{x}{a} > 0\right)$$

523. $$\int \tanh^{-1}\frac{x}{a}\,dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\log_e(a^2-x^2)$$

524. $$\int x\tanh^{-1}\frac{x}{a}\,dx = \frac{x^2-a^2}{2}\tanh^{-1}\frac{x}{a} + \frac{ax}{2}$$

525. $$\int \coth^{-1}\frac{x}{a}\,dx = x\coth^{-1}\frac{x}{a} + \frac{a}{2}\log_e(x^2-a^2)$$