

$$1. \int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$2. \int \frac{dx}{x} = \log_e |x| \quad (x \neq 0)$$

$$3. \int \sin x dx = -\cos x$$

$$4. \int \cos x dx = \sin x$$

$$5. \int \tan x dx = -\log_e |\cos x|$$

$$6. \int \cot x dx = \log_e |\sin x|$$

$$7. \int \frac{dx}{\cos^2 x} = \tan x$$

$$8. \int \frac{dx}{\sin^2 x} = -\cot x$$

$$9. \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} \quad (a \neq 0)$$

$$10. \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \tanh^{-1} \frac{x}{a} = \frac{1}{2a} \log_e \frac{a+x}{a-x} \quad (|x| < a)$$

$$11. \int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \coth^{-1} \frac{x}{a} = \frac{1}{2a} \log_e \frac{x-a}{x+a} \quad (|x| > a)$$

$$12. \int e^x dx = e^x$$

$$13. \int a^x dx = \frac{a^x}{\log_e a} \quad (a > 0, a \neq 1)$$

$$14. \int \sinh x dx = \cosh x$$

$$15. \int \cosh x dx = \sinh x$$

$$16. \int \tanh x dx = \log_e \cosh x$$

$$17. \int \coth x dx = \log_e |\sinh x|$$

$$18. \int \frac{dx}{\cosh^2 x} = \tanh x$$

$$19. \int \frac{dx}{\sinh^2 x} = -\coth x$$

$$20. \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} \quad (a \neq 0)$$

$$21. \int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a} = \log_e |x + \sqrt{a^2 + x^2}| + C_1$$

$$22. \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} = \log_e |x + \sqrt{x^2 - a^2}| + C_1$$

(a) Integrals containing $ax + b$ ($a \neq 0$)

$$1. \int (ax + b)^n dx = \frac{1}{a(n+1)}(ax + b)^{n+1} \quad (n \neq -1)$$

$$2. \int \frac{dx}{ax + b} = \frac{1}{a} \log_e(ax + b)$$

$$3. \int x(ax + b)^n dx = \frac{1}{a^2(n+2)}(ax + b)^{n+2}$$

$$- \frac{b}{a^2(n+1)}(ax + b)^{n+1} \quad (n \neq -1, -2)$$

$$4. \int x^m(ax + b)^n dx$$

$$= \frac{1}{a(m+n+1)} \left[x^m(ax + b)^{n+1} - mb \int x^{m-1}(ax + b)^n dx \right]$$

$$= \frac{1}{m+n+1} \left[x^{m+1}(ax + b)^n + nb \int x^m(ax + b)^{n-1} dx \right]$$

$$(m > 0, m+n+1 \neq 0)$$

$$5. \int \frac{x dx}{ax + b} = \frac{x}{a} - \frac{b}{a^2} \log_e(ax + b)$$

$$6. \int \frac{x dx}{(ax + b)^2} = \frac{b}{a^2(ax + b)} + \frac{1}{a^2} \log_e(ax + b)$$

$$7. \int \frac{x dx}{(ax + b)^3} = \frac{b}{2a^2(ax + b)^2} - \frac{1}{a^2(ax + b)}$$

$$8. \int \frac{x dx}{(ax + b)^n} = \frac{1}{a^2} \left[\frac{b}{(n-1)(ax + b)^{n-1}} - \frac{1}{(n-2)(ax + b)^{n-2}} \right]$$

$$(n \neq 1, 2)$$

$$9. \int \frac{x^2 dx}{ax+b} = \frac{1}{a^3} \left[\frac{1}{2} (ax+b)^2 - 2b(ax+b) + b^2 \log_e (ax+b) \right]$$

$$10. \int \frac{x^2 dx}{(ax+b)^2} = \frac{1}{a^3} \left[(ax+b) - 2b \log_e (ax+b) - \frac{b^2}{ax+b} \right]$$

$$11. \int \frac{x^2 dx}{(ax+b)^3} = \frac{1}{a^3} \left[\log_e (ax+b) + \frac{2b}{ax+b} - \frac{b^2}{2(ax+b)^2} \right]$$

$$12. \int x^2(ax+b)^n dx$$

$$= \frac{1}{a^3} \left[\frac{(ax+b)^{n+3}}{n+3} - 2b \frac{(ax+b)^{n+2}}{n+2} + b^2 \frac{(ax+b)^{n+1}}{n+1} \right]$$

$$(n \neq -1, -2, -3)$$

$$13. \int \frac{dx}{x(ax+b)} = \frac{1}{b} \log_e \frac{x}{ax+b}$$

$$14. \int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} - \frac{1}{b^2} \log_e \frac{ax+b}{x}$$

$$15. \int \frac{dx}{x(ax+b)^3} = \frac{1}{b^3} \left[\frac{1}{2} \left(\frac{ax+2b}{ax+b} \right)^2 + \log_e \frac{x}{ax+b} \right]$$

$$16. \int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2} \log_e \frac{ax+b}{x}$$

$$17. \int \frac{dx}{x^2(ax+b)^2} = -\frac{b+2ax}{b^2x(ax+b)} + \frac{2a}{b^3} \log_e \frac{ax+b}{x}$$

$$18. \int \frac{dx}{x^3(ax+b)} = \frac{2ax-b}{2b^2x^2} + \frac{a^2}{b^3} \log_e \frac{x}{ax+b}$$

Let $ax+b \equiv X$ ($a \neq 0$). Then

$$19. \int \frac{x^3 dx}{X} = \frac{1}{a^4} \left(\frac{X^3}{3} - \frac{3bX^2}{2} + 3b^2X - b^3 \log_e X \right)$$

$$20. \int \frac{x^3 dx}{X^2} = \frac{1}{a^4} \left(\frac{X^2}{2} - 3bX + 3b^2 \log_e X + \frac{b^3}{X} \right)$$

$$21. \int \frac{x^3 dx}{X^3} = \frac{1}{a^4} \left(X - 3b \log_e X - \frac{3b^2}{X} + \frac{b^3}{2X^2} \right)$$

$$22. \int \frac{x^3 dx}{X^4} = \frac{1}{a^4} \left(\log_e X + \frac{3b}{X} - \frac{3b^2}{2X^2} + \frac{b^3}{3X^3} \right)$$

$$23. \int \frac{x^3 dx}{X^n} = \frac{1}{a^4} \left[\frac{-1}{(n-4)X^{n-4}} + \frac{3b}{(n-3)X^{n-3}} - \frac{3b^2}{(n-2)X^{n-2}} + \frac{b^3}{(n-1)X^{n-1}} \right] \quad (n \neq 1, 2, 3, 4)$$

$$24. \int \frac{dx}{xX^n} = -\frac{1}{b^n} \left[\log_e \frac{X}{x} - \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{(-a)^k x^k}{kX^k} \right], \quad (n \geq 1)$$

$$25. \int \frac{dx}{x^2 X^3} = -a \left[\frac{1}{2b^2 X^2} + \frac{2}{b^3 X} + \frac{1}{ab^3 x} - \frac{3}{b^4} \log_e \frac{X}{x} \right]$$

$$26. \int \frac{dx}{x^2 X^n} = -\frac{1}{b^{n+1}} \left[-\sum_{k=2}^n \binom{n}{k} \frac{(-a)^k x^{k-1}}{(k-1)X^{k-1}} + \frac{X}{x} - na \log_e \frac{X}{x} \right] \quad (n \geq 2)$$

$$27. \int \frac{dx}{x^3 X^2} = -\frac{1}{b^4} \left[3a^2 \log_e \frac{X}{x} + \frac{a^3 x}{X} + \frac{X^2}{2x^2} - \frac{3aX}{x} \right]$$

$$28. \int \frac{dx}{x^3 X^3} = -\frac{1}{b^5} \left[6a^2 \log_e \frac{X}{x} + \frac{4a^3 x}{X} + \frac{4a^2 x}{X} - \frac{a^4 x^2}{2X^2} + \frac{X^2}{2x^2} - \frac{4aX}{x} \right]$$

$$29. \int \frac{dx}{x^3 X^n} = -\frac{1}{b^{n+2}} \left[-\sum_{k=3}^{n+1} \binom{n+1}{k} \frac{(-a)^k x^{k-2}}{(k-2)X^{k-2}} + \frac{a^2 X^2}{2x^2} - \frac{(n+1)aX}{x} + \frac{n(n+1)a^2}{2} \log_e \frac{X}{x} \right] \quad (n \geq 3)$$

$$30. \int \frac{dx}{x^m X^n} = -\frac{1}{b^{m+n-1}} \sum_{k=0}^{m+n-2} \binom{m+n-2}{k} \frac{X^{m-k-1} (-a)^k}{(m-k-1)x^{m-k-1}}$$

[terms with $(m-k-1) = 0$ are replaced by

$$\binom{m+n-2}{m-1} (-a)^{m-1} \log_e \frac{X}{x}]$$

(b) Integrals containing $ax + b$ and $cx + d$ ($a \neq 0, c \neq 0$)

$$31. \int \frac{ax + b}{cx + d} dx = \frac{a}{c} x + \frac{bc - ad}{c^2} \log_e (cx + d)$$

$$32. \int \frac{dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad} \log_e \frac{cx+d}{ax+b} \quad (bc-ad \neq 0)$$

$$33. \int \frac{x dx}{(ax+b)(cx+d)}$$

$$= \frac{1}{bc-ad} \left[\frac{b}{a} \log(ax+b) - \frac{d}{c} \log_e(cx+d) \right] \quad (bc-ad \neq 0)$$

$$34. \int \frac{dx}{(ax+b)^2(cx+d)}$$

$$= \frac{1}{bc-ad} \left[\frac{1}{ax+b} + \frac{c}{bc-ad} \log_e \frac{cx+d}{ax+b} \right] \quad (bc-ad \neq 0)$$

(c) Integrals containing $a+x$ and $b+x$ ($a \neq b$)

$$35. \int \frac{x dx}{(a+x)(b+x)^2} = \frac{b}{(a-b)(b+x)} - \frac{a}{(a-b)^2} \log_e \frac{a+x}{b+x}$$

$$36. \int \frac{x^2 dx}{(a+x)(b+x)^2} = \frac{b^2}{(b-a)(b+x)} + \frac{a^2}{(b-a)^2} \log_e(a+x) + \frac{b^2 - 2ab}{(b-a)^2} \log_e(b+x)$$

$$37. \int \frac{dx}{(a+x)^2(b+x)^2}$$

$$= \frac{-1}{(a-b)^2} \left(\frac{1}{a+x} + \frac{1}{b+x} \right) + \frac{2}{(a-b)^3} \log_e \frac{a+x}{b+x}$$

$$38. \int \frac{x dx}{(a+x)^2(b+x)^2}$$

$$= \frac{1}{(a-b)^2} \left(\frac{a}{a+x} + \frac{b}{b+x} \right) + \frac{a+b}{(a-b)^3} \log_e \frac{a+x}{b+x}$$

$$39. \int \frac{x^2 dx}{(a+x)^2(b+x)^2}$$

$$= \frac{-1}{(a-b)^2} \left(\frac{a^2}{a+x} + \frac{b^2}{b+x} \right) + \frac{2ab}{(a-b)^3} \log_e \frac{a+x}{b+x}$$

(d) Integrals containing $ax^2 + bx + c$ ($a \neq 0$)

$$40. \int \frac{dx}{ax^2 + bx + c}$$

$$= \begin{cases} \frac{1}{\sqrt{b^2 - 4ac}} \log_e \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} & (b^2 > 4ac) \\ \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}} & (b^2 < 4ac) \\ -\frac{2}{2ax + b} & (b^2 = 4ac) \end{cases}$$

In numbers 41 through 47, let $b^2 - 4ac \neq 0$.

$$41. \int \frac{dx}{(ax^2 + bx + c)^2} = \frac{2ax + b}{(4ac - b^2)(ax^2 + bx + c)} + \frac{2a}{(4ac - b^2)} \int \frac{dx}{ax^2 + bx + c}$$

$$42. \int \frac{dx}{(ax^2 + bx + c)^{n+1}} = \frac{2ax + b}{n(4ac - b^2)(ax^2 + bx + c)^n} + \frac{2(2n - 1)a}{n(4ac - b^2)} \int \frac{dx}{(ax^2 + bx + c)^n}$$

$$43. \int \frac{x dx}{ax^2 + bx + c} = \frac{1}{2a} \log_e (ax^2 + bx + c) - \frac{b}{2a} \int \frac{dx}{ax^2 + bx + c}$$

$$44. \int \frac{x^2 dx}{ax^2 + bx + c} = \frac{x}{a} - \frac{b}{2a^2} \log_e (ax^2 + bx + c) + \frac{b^2 - 2ac}{2a^2} \int \frac{dx}{ax^2 + bx + c}$$

$$45. \int \frac{x^n dx}{ax^2 + bx + c} = \frac{x^{n-1}}{(n-1)a} - \frac{c}{a} \int \frac{x^{n-2} dx}{ax^2 + bx + c} - \frac{b}{a} \int \frac{x^{n-1} dx}{ax^2 + bx + c} \quad (n \neq 1)$$

$$46. \int \frac{x dx}{(ax^2 + bx + c)^{n+1}} = \frac{-2c + bx}{n(4ac - b^2)(ax^2 + bx + c)^n} - \frac{b(2n - 1)}{n(4ac - b^2)} \int \frac{dx}{(ax^2 + bx + c)^n}$$

$$47. \int \frac{x^m dx}{(ax^2 + bx + c)^{n+1}} = -\frac{x^{m-1}}{a(2n-m+1)(ax^2 + bx + c)^n} \\ - \frac{n-m+1}{2n-m+1} \cdot \frac{b}{a} \int \frac{x^{m-1} dx}{(ax^2 + bx + c)^{n+1}} \\ + \frac{m-1}{2n-m+1} \cdot \frac{c}{a} \int \frac{x^{m-2} dx}{(ax^2 + bx + c)^{n+1}} \quad (m \neq 2n+1)$$

Let $ax^2 + bx + c \equiv X$ ($a \neq 0$). Then

$$48. \int \frac{x^{2n-1} dx}{X^n} = \frac{1}{a} \int \frac{x^{2n-3} dx}{X^{n-1}} - \frac{c}{a} \int \frac{x^{2n-3} dx}{X^n} - \frac{b}{a} \int \frac{x^{2n-2} dx}{X^n}$$

$$49. \int \frac{dx}{xX} = \frac{1}{2c} \log_e \frac{x^2}{X} - \frac{b}{2c} \int \frac{dx}{X}$$

$$50. \int \frac{dx}{xX^n} = \frac{1}{2c(n-1)X^{n-1}} - \frac{b}{2c} \int \frac{dx}{X^n} + \frac{1}{c} \int \frac{dx}{xX^{n-1}}$$

$$51. \int \frac{dx}{x^2 X} = \frac{b}{2c^2} \log_e \frac{X}{x^2} - \frac{1}{cx} + \left(\frac{b^2}{2c^2} - \frac{a}{c} \right) \int \frac{dx}{X}$$

$$52. \int \frac{dx}{x^m X^n} = -\frac{1}{(m-1)cx^{m-1}X^{n-1}} - \frac{(2n+m-3)a}{(m-1)c} \int \frac{dx}{x^{m-2}X^n} \\ - \frac{(n+m-2)b}{(m-1)c} \int \frac{dx}{x^{m-1}X^n} \quad (m > 1)$$

$$53. \int \frac{dx}{(fx+g)X} = \frac{1}{2(cf^2 - gbf + g^2a)} \left[f \log_e \frac{(fx+g)^2}{X} \right]$$

$$+ \frac{2ga - bf}{2(cf^2 - gbf + g^2a)} \int \frac{dx}{X}$$

(e) Integrals containing $a^2 \pm x^2$, with

$$X \equiv a^2 + x^2 \quad Y \equiv \arctan \frac{x}{a}$$

or

$$X \equiv a^2 - x^2 \quad Y \equiv \tanh^{-1} \frac{x}{a} \equiv \begin{cases} \frac{1}{2} \log_e \frac{a+x}{a-x} & (|x| < a) \\ \frac{1}{2} \log_e \frac{x+a}{x-a} & (|x| > a) \end{cases}$$

Where \pm or \mp appears in a formula, the upper sign refers to $X \equiv a^2 + x^2$ and the lower sign to $X \equiv a^2 - x^2$ ($a \neq 0$).

$$54. \int \frac{dx}{X} = \frac{1}{a} Y$$

$$55. \int \frac{dx}{X^2} = \frac{x}{2a^2 X} + \frac{1}{2a^3} Y$$

$$56. \int \frac{dx}{X^3} = \frac{x}{4a^2 X^2} + \frac{3x}{8a^4 X} + \frac{3}{8a^5} Y$$

$$57. \int \frac{dx}{X^{n+1}} = \frac{x}{2na^2 X^n} + \frac{2n-1}{2na^2} \int \frac{dx}{X^n}$$

$$58. \int \frac{x dx}{X} = \pm \frac{1}{2} \log_e X$$

$$59. \int \frac{x dx}{X^2} = \mp \frac{1}{2X}$$

$$60. \int \frac{x dx}{X^3} = \mp \frac{1}{4X^2}$$

$$61. \int \frac{x dx}{X^{n+1}} = \mp \frac{1}{2nX^n} \quad (n \neq 0)$$

$$62. \int \frac{x^2 dx}{X} = \pm x \mp aY$$

$$63. \int \frac{x^2 dx}{X^2} = \mp \frac{x}{2X} \pm \frac{1}{2a} Y$$

$$64. \int \frac{x^2 dx}{X^3} = \mp \frac{x}{4X^2} \pm \frac{x}{8a^2 X} \pm \frac{1}{8a^3} Y$$

$$65. \int \frac{x^2 dx}{X^{n+1}} = \mp \frac{x}{2nX^n} \pm \frac{1}{2n} \int \frac{dx}{X^n} \quad (n \neq 0)$$

$$66. \int \frac{x^3 dx}{X} = \pm \frac{x^2}{2} - \frac{a^2}{2} \log_e X$$

$$67. \int \frac{x^3 dx}{X^2} = \frac{a^2}{2X} + \frac{1}{2} \log_e X$$

$$68. \int \frac{x^3 dx}{X^3} = -\frac{1}{2X} + \frac{a^2}{4X^2}$$

$$69. \int \frac{x^3 dx}{X^{n+1}} = -\frac{1}{2(n-1)X^{n-1}} + \frac{a^2}{2nX^n} \quad (n > 1)$$

$$70. \int \frac{dx}{x\bar{X}} = \frac{1}{2a^2} \log_e \bar{X}$$

$$71. \int \frac{dx}{xX^2} = \frac{1}{2a^2\bar{X}} + \frac{1}{2a^4} \log_e \bar{X}$$

$$72. \int \frac{dx}{x\bar{X}^3} = \frac{1}{4a^2X^2} + \frac{1}{2a^4\bar{X}} + \frac{1}{2a^6} \log_e \bar{X}$$

$$73. \int \frac{dx}{x^2\bar{X}} = -\frac{1}{a^2x} \mp \frac{1}{a^3} Y$$

$$74. \int \frac{dx}{x^2\bar{X}^2} = -\frac{1}{a^4x} \mp \frac{x}{2a^4\bar{X}} \mp \frac{3}{2a^5} Y$$

$$75. \int \frac{dx}{x^2\bar{X}^3} = -\frac{1}{a^6x} \mp \frac{x}{4a^4X^2} \mp \frac{7x}{8a^6\bar{X}} \mp \frac{15}{8a^7} Y$$

$$76. \int \frac{dx}{x^3\bar{X}} = -\frac{1}{2a^2x^2} \mp \frac{1}{2a^4} \log_e \bar{X}$$

$$77. \int \frac{dx}{x^3\bar{X}^2} = -\frac{1}{2a^4x^2} \mp \frac{1}{2a^4\bar{X}} \mp \frac{1}{a^6} \log_e \bar{X}$$

$$78. \int \frac{dx}{x^3\bar{X}^3} = -\frac{1}{2a^6x^2} \mp \frac{1}{a^6\bar{X}} \mp \frac{1}{4a^4X^2} \mp \frac{3}{2a^8} \log_e \bar{X}$$

$$79. \int \frac{dx}{(b+cx)X} = \frac{1}{a^2c^2 \pm b^2} \left[c \log_e (b+cx) - \frac{c}{2} \log_e X \pm \frac{b}{a} Y \right]$$

(f) Integrals containing $a^3 \pm x^3$, with

$$X \equiv a^3 \pm x^3 \quad (a \neq 0)$$

Where \pm or \mp appears in a formula, the upper sign refers to $X \equiv a^3 + x^3$ and the lower sign to $X \equiv a^3 - x^3$.

$$80. \int \frac{dx}{X} = \pm \frac{1}{6a^2} \log_e \frac{(a \pm x)^2}{a^2 \mp ax + x^2} + \frac{1}{a^2\sqrt{3}} \arctan \frac{2x \mp a}{a\sqrt{3}}$$

$$81. \int \frac{dx}{X^2} = \frac{x}{3a^3X} + \frac{2}{3a^3} \int \frac{dx}{X}$$

$$92. \int \frac{x dx}{X} = \frac{1}{6a} \log_e \frac{a^2 \mp ax + x^2}{(a \pm x)^2} \pm \frac{1}{a\sqrt{3}} \arctan \frac{2x \mp a}{a\sqrt{3}}$$

$$83. \int \frac{x dx}{X^2} = \frac{x^2}{3a^2 X} + \frac{1}{3a^3} \int \frac{x dx}{X}$$

$$84. \int \frac{x^2 dx}{X} = \pm \frac{1}{3} \log_e X$$

$$85. \int \frac{x^2 dx}{X^2} = \mp \frac{1}{3X}$$

$$86. \int \frac{x^3 dx}{X} = \pm x \mp a^3 \int \frac{dx}{X}$$

$$87. \int \frac{x^3 dx}{X^2} = \mp \frac{x}{3X} \pm \frac{1}{3} \int \frac{dx}{X}$$

$$88. \int \frac{dx}{xX} = \frac{1}{3a^3} \log_e \frac{x^2}{X}$$

$$89. \int \frac{dx}{xX^2} = \frac{1}{3a^3 X} + \frac{1}{3a^6} \log_e \frac{x^2}{X}$$

$$90. \int \frac{dx}{x^2 X} = -\frac{1}{a^2 x} \mp \frac{1}{a^3} \int \frac{x dx}{X}$$

$$91. \int \frac{dx}{x^2 X^2} = -\frac{1}{a^6 x} \mp \frac{x^2}{3a^6 X} \mp \frac{4}{3a^6} \int \frac{x dx}{X}$$

$$92. \int \frac{dx}{x^3 X} = -\frac{1}{2a^2 x^2} \mp \frac{1}{a^3} \int \frac{dx}{X}$$

$$93. \int \frac{dx}{x^3 X^2} = -\frac{1}{2a^5 x^2} \mp \frac{x}{3a^6 X} \mp \frac{5}{3a^6} \int \frac{dx}{X}$$

(g) Integrals containing $a^4 \pm x^4$ ($a \neq 0$)

$$94. \int \frac{dx}{a^4 + x^4} = \frac{1}{4a^3 \sqrt{2}} \log_e \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a^3 \sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2 - x^2}$$

$$95. \int \frac{x dx}{a^4 + x^4} = \frac{1}{2a^2} \arctan \frac{x^2}{a^2}$$

$$96. \int \frac{x^2 dx}{a^4 + x^4} = -\frac{1}{4a\sqrt{2}} \log_e \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2 - x^2}$$

$$97. \int \frac{x^3 dx}{a^4 + x^4} = \frac{1}{4} \log_e (a^4 + x^4)$$

$$98. \int \frac{dx}{a^4 - x^4} = \frac{1}{4a^3} \log_e \frac{a+x}{a-x} + \frac{1}{2a^3} \arctan \frac{x}{a}$$

$$99. \int \frac{x dx}{a^4 - x^4} = \frac{1}{4a^3} \log_e \frac{a^2 + x^2}{a^2 - x^2}$$

$$100. \int \frac{x^2 dx}{a^4 - x^4} = \frac{1}{4a} \log_e \frac{a+x}{a-x} - \frac{1}{2a} \arctan \frac{x}{a}$$

$$101. \int \frac{x^3 dx}{a^4 - x^4} = -\frac{1}{4} \log_e (a^4 - x^4)$$

(h) Integrals containing \sqrt{x} and $a^2 + b^2x$ ($a, b \neq 0$)

$$102. \int \frac{\sqrt{x} dx}{a^2 + b^2x} = \frac{2\sqrt{x}}{b^2} - \frac{2a}{b^3} \arctan \frac{b\sqrt{x}}{a}$$

$$103. \int \frac{x\sqrt{x} dx}{a^2 + b^2x} = \frac{2x\sqrt{x}}{3b^2} - \frac{2a^2\sqrt{x}}{b^4} + \frac{2a^3}{b^5} \arctan \frac{b\sqrt{x}}{a}$$

$$104. \int \frac{\sqrt{x} dx}{(a^2 + b^2x)^2} = -\frac{\sqrt{x}}{b^2(a^2 + b^2x)} + \frac{1}{ab^3} \arctan \frac{b\sqrt{x}}{a}$$

$$105. \int \frac{x\sqrt{x} dx}{(a^2 + b^2x)^2} = \frac{2b^2x\sqrt{x} + 3a^2\sqrt{x}}{b^4(a^2 + b^2x)} - \frac{3a}{b^5} \arctan \frac{b\sqrt{x}}{a}$$

$$106. \int \frac{dx}{(a^2 + b^2x)\sqrt{x}} = \frac{2}{ab} \arctan \frac{b\sqrt{x}}{a}$$

$$107. \int \frac{dx}{(a^2 + b^2x)x\sqrt{x}} = -\frac{e^2}{a^2\sqrt{x}} - \frac{2b}{a^3} \arctan \frac{b\sqrt{x}}{a}$$

$$108. \int \frac{dx}{(a^2 + b^2x)^2\sqrt{x}} = \frac{\sqrt{x}}{a^2(a^2 + b^2x)} + \frac{1}{a^3b} \arctan \frac{b\sqrt{x}}{a}$$

(i) Integrals containing \sqrt{x} and $a^2 - b^2x > 0$ ($a, b \neq 0$)

$$109. \int \frac{\sqrt{x} dx}{a^2 - b^2x} = -\frac{2\sqrt{x}}{b^2} + \frac{a}{b^3} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$$

$$110. \int \frac{x\sqrt{x} dx}{a^2 - b^2x} = -\frac{2x\sqrt{x}}{3b^2} - \frac{2a^2\sqrt{x}}{b^4} + \frac{a^3}{b^5} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$$

$$111. \int \frac{\sqrt{x} dx}{(a^2 - b^2x)^2} = \frac{\sqrt{x}}{b^2(a^2 - b^2x)} - \frac{1}{2ab^3} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$$

$$112. \int \frac{x\sqrt{x} dx}{(a^2 - b^2x)^2} = \frac{-2b^2x\sqrt{x} + 3a^2\sqrt{x}}{b^4(a^2 - b^2x)} - \frac{3a}{2b^5} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$$

$$113. \int \frac{dx}{(a^2 - b^2x)\sqrt{x}} = \frac{1}{ab} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$$

$$114. \int \frac{dx}{(a^2 - b^2x)x\sqrt{x}} = -\frac{2}{a^2\sqrt{x}} + \frac{b}{a^3} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$$

$$115. \int \frac{dx}{(a^2 - b^2x)^2\sqrt{x}} = \frac{\sqrt{x}}{a^2(a^2 - b^2x)} + \frac{1}{2a^3b} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$$

$$116. \int \frac{dx}{(a^2 - b^2x)^2x\sqrt{x}} = -\frac{2}{a^2(a^2 - b^2x)\sqrt{x}} + \frac{3b^2\sqrt{x}}{a^4(a^2 - b^2x)}$$

$$+ \frac{3b}{2a^5} \log_e \frac{a + b\sqrt{x}}{a - b\sqrt{x}}$$

(j) *Other integrals containing* \sqrt{x} ($a > \sqrt{x} > 0$)

$$117. \int \frac{\sqrt{x} dx}{a^4 + x^2} = -\frac{1}{2a\sqrt{2}} \log_e \frac{x + a\sqrt{2x} + a^2}{x - a\sqrt{2x} + a^2} + \frac{1}{a\sqrt{2}} \arctan \frac{a\sqrt{2x}}{a^2 - x}$$

$$118. \int \frac{dx}{(a^4 + x^2)\sqrt{x}} = \frac{1}{2a^3\sqrt{2}} \log_e \frac{x + a\sqrt{2x} + a^2}{x - a\sqrt{2x} + a^2} + \frac{1}{a^3\sqrt{2}} \arctan \frac{a\sqrt{2x}}{a^2 - x}$$

$$119. \int \frac{\sqrt{x} dx}{a^4 - x^2} = \frac{1}{2a} \log_e \frac{a + \sqrt{x}}{a - \sqrt{x}} - \frac{1}{a} \arctan \frac{\sqrt{x}}{a}$$

$$120. \int \frac{dx}{(a^4 - x^2)\sqrt{x}} = \frac{1}{2a^3} \log_e \frac{a + \sqrt{x}}{a - \sqrt{x}} + \frac{1}{a^3} \arctan \frac{\sqrt{x}}{a}$$

(k) *Integrals containing* $\sqrt{ax + b}$, with

$$X \equiv ax + b \quad Y \equiv fx + g \quad \Delta \equiv bf - ag \quad (a \neq 0)$$

121. $\int \sqrt{X} dx = \frac{2}{3a} \sqrt{X^3}$
122. $\int x\sqrt{X} dx = \frac{2(3ax - 2b)\sqrt{X^3}}{15a^2}$
123. $\int x^2\sqrt{X} dx = \frac{2(15a^2x^2 - 12abx + 8b^2)\sqrt{X^3}}{105a^3}$
124. $\int \frac{dx}{\sqrt{X}} = \frac{2\sqrt{X}}{a}$
125. $\int \frac{x dx}{\sqrt{X}} = \frac{2(ax - 2b)}{3a^2} \sqrt{X}$
126. $\int \frac{x^2 dx}{\sqrt{X}} = \frac{2(3a^2x^2 - 4abx + 8b^2)\sqrt{X}}{15a^3}$
127. $\int \frac{dx}{x\sqrt{X}} = \begin{cases} -\frac{2}{\sqrt{b}} \tanh^{-1} \sqrt{\frac{X}{b}} = \frac{1}{\sqrt{b}} \log_e \frac{\sqrt{X} - \sqrt{b}}{\sqrt{X} + \sqrt{b}} & (b > 0) \\ \frac{2}{\sqrt{-b}} \arctan \sqrt{\frac{X}{-b}} & (b < 0) \end{cases}$
128. $\int \frac{\sqrt{X}}{x} dx = 2\sqrt{X} + b \int \frac{dx}{x\sqrt{X}}$
129. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{bx} - \frac{a}{2b} \int \frac{dx}{x\sqrt{X}}$
130. $\int \frac{\sqrt{X}}{x^2} dx = -\frac{\sqrt{X}}{x} + \frac{a}{2} \int \frac{dx}{x\sqrt{X}}$
131. $\int \frac{dx}{x^n\sqrt{X}} = -\frac{\sqrt{X}}{(n-1)b x^{n-1}} - \frac{(2n-3)a}{(2n-2)b} \int \frac{dx}{x^{n-1}\sqrt{X}}$ ($n \neq 1$)
132. $\int \sqrt{X^3} dx = \frac{2\sqrt{X^6}}{5a}$
133. $\int x\sqrt{X^3} dx = \frac{2}{35a^2} (5\sqrt{X^7} - 7b\sqrt{X^5})$
134. $\int x^2\sqrt{X^3} dx = \frac{2}{a^3} \left(\frac{\sqrt{X^9}}{9} - \frac{2b\sqrt{X^7}}{7} + \frac{b^2\sqrt{X^5}}{5} \right)$

$$135. \int \frac{\sqrt{X^3}}{x} dx = \frac{2\sqrt{X^3}}{3} + 2b\sqrt{X} + b^2 \int \frac{dx}{x\sqrt{X}}$$

$$136. \int \frac{x dx}{\sqrt{X^3}} = \frac{2}{a^2} \left(\sqrt{X} + \frac{b}{\sqrt{X}} \right)$$

$$137. \int \frac{x^2 dx}{\sqrt{X^3}} = \frac{2}{a^3} \left(\frac{\sqrt{X^3}}{3} - 2b\sqrt{X} - \frac{b^2}{\sqrt{X}} \right)$$

$$138. \int \frac{dx}{x\sqrt{X^3}} = \frac{2}{b\sqrt{X}} + \frac{1}{b} \int \frac{dx}{x\sqrt{X}} \quad (b \neq 0)$$

$$139. \int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{bx\sqrt{X}} - \frac{3a}{b^2\sqrt{X}} - \frac{3a}{2b^2} \int \frac{dx}{x\sqrt{X}} \quad (b \neq 0)$$

$$140. \int X^{\pm n/2} dx = \frac{2X^{(2\pm n)/2}}{a(2 \pm n)} \quad (n \neq \mp 2)$$

$$141. \int xX^{\pm n/2} dx = \frac{2}{a^2} \left(\frac{X^{(4\pm n)/2}}{4 \pm n} - \frac{bX^{(2\pm n)/2}}{2 \pm n} \right) \quad (n \neq \mp 2, \mp 4)$$

$$142. \int x^2X^{\pm n/2} dx = \frac{2}{a^3} \left(\frac{X^{(6\pm n)/2}}{6 \pm n} - \frac{2bX^{(4\pm n)/2}}{4 \pm n} + \frac{b^2X^{(2\pm n)/2}}{2 \pm n} \right) \quad (n \neq \mp 2, \mp 4, \mp 6)$$

$$143. \int \frac{X^{n/2} dx}{x} = \frac{2X^{n/2}}{n} + b \int \frac{X^{(n-2)/2}}{x} dx \quad (n \neq 0)$$

$$144. \int \frac{dx}{xX^{n/2}} = \frac{2}{(n-2)bX^{(n-2)/2}} + \frac{1}{b} \int \frac{dx}{xX^{(n-2)/2}} \quad (n \neq 2, b \neq 0)$$

$$145. \int \frac{dx}{x^2X^{n/2}} = -\frac{1}{bxX^{(n-2)/2}} - \frac{na}{2b} \int \frac{dx}{xX^{n/2}} \quad (b \neq 0)$$

$$146. \int \frac{dx}{\sqrt{XY}} = \begin{cases} \frac{2}{\sqrt{-af}} \arctan \sqrt{\frac{fX}{-aY}} & (af < 0) \\ \frac{2}{\sqrt{af}} \tanh^{-1} \sqrt{\frac{fX}{aY}} = \frac{2}{\sqrt{af}} \log_e (\sqrt{aY} + \sqrt{fX}) + C_1 & (af > 0) \end{cases}$$

$$147. \int \frac{x dx}{\sqrt{XY}} = \frac{\sqrt{XY}}{af} - \frac{ag + bf}{2af} \int \frac{dx}{\sqrt{XY}}$$

$$148. \int \frac{dx}{\sqrt{X}\sqrt{Y^3}} = -\frac{2\sqrt{X}}{\Delta\sqrt{Y}}$$

$$149. \int \frac{dx}{Y\sqrt{X}} = \begin{cases} \frac{2}{\sqrt{-f\Delta}} \arctan \frac{f\sqrt{X}}{\sqrt{-f\Delta}} & (f\Delta < 0) \\ \frac{1}{\sqrt{f\Delta}} \log_e \frac{f\sqrt{X} - \sqrt{f\Delta}}{f\sqrt{X} + \sqrt{f\Delta}} & (f\Delta > 0) \end{cases}$$

$$150. \int \sqrt{XY} dx = \frac{\Delta + 2aY}{4af} \sqrt{XY} - \frac{\Delta^2}{8af} \int \frac{dx}{\sqrt{XY}}$$

$$151. \int \frac{Y}{\sqrt{X}} dx = \frac{1}{a} \sqrt{XY} - \frac{\Delta}{2a} \int \frac{dx}{\sqrt{XY}}$$

$$152. \int \frac{\sqrt{X} dx}{Y} = \frac{2\sqrt{X}}{f} + \frac{\Delta}{f} \int \frac{dx}{Y\sqrt{X}} \quad (f \neq 0)$$

$$153. \int \frac{Y^n dx}{\sqrt{X}} = \frac{2}{(2n+1)a} \left(\sqrt{X}Y^n - n\Delta \int \frac{Y^{n-1} dx}{\sqrt{X}} \right)$$

$$154. \int \frac{dx}{\sqrt{X}Y^n} = -\frac{1}{(n-1)\Delta} \left\{ \frac{\sqrt{X}}{Y^{n-1}} + \left(n - \frac{3}{2}\right)a \int \frac{dx}{\sqrt{X}Y^{n-1}} \right\} \quad (\Delta \neq 0, n \neq 1)$$

$$155. \int \sqrt{X}Y^n dx = \frac{1}{(2n+3)f} \left(2\sqrt{X}Y^{n+1} + \Delta \int \frac{Y^n dx}{\sqrt{X}} \right) \quad (f \neq 0)$$

$$156. \int \frac{\sqrt{X} dx}{Y^n} = \frac{1}{(n-1)f} \left(-\frac{\sqrt{X}}{Y^{n-1}} + \frac{a}{2} \int \frac{dx}{\sqrt{X}Y^{n-1}} \right) \quad (f \neq 0, n \neq 1)$$

(1) Integrals containing $\sqrt{a^2 - x^2}$, with

$$\boxed{\sqrt{X} \equiv a^2 - x^2 \quad (a > 0)}$$

$$157. \int \sqrt{X} dx = \frac{1}{2} \left(x\sqrt{X} + a^2 \arcsin \frac{x}{a} \right)$$

$$158. \int x\sqrt{X} dx = -\frac{1}{3}\sqrt{X^3}$$

$$159. \int x^2 \sqrt{X} dx = -\frac{x}{4} \sqrt{X^3} + \frac{a^2}{8} \left(x \sqrt{X} + a^2 \arcsin \frac{x}{a} \right)$$

$$160. \int x^3 \sqrt{X} dx = \frac{\sqrt{X^5}}{5} - a^2 \frac{\sqrt{X^3}}{3}$$

$$161. \int \frac{\sqrt{X}}{x} dx = \sqrt{X} - a \log_e \frac{a + \sqrt{X}}{x}$$

$$162. \int \frac{\sqrt{X}}{x^2} dx = -\frac{\sqrt{X}}{x} - \arcsin \frac{x}{a}$$

$$163. \int \frac{\sqrt{X}}{x^3} dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a} \log_e \frac{a + \sqrt{X}}{x}$$

$$164. \int \frac{dx}{\sqrt{X}} = \arcsin \frac{x}{a}$$

$$165. \int \frac{x dx}{\sqrt{X}} = -\sqrt{X}$$

$$166. \int \frac{x^2 dx}{\sqrt{X}} = -\frac{x}{2} \sqrt{X} + \frac{a^2}{2} \arcsin \frac{x}{a}$$

$$167. \int \frac{x^3 dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} - a^2 \sqrt{X}$$

$$168. \int \frac{dx}{x \sqrt{X}} = -\frac{1}{a} \log_e \frac{a + \sqrt{X}}{x}$$

$$169. \int \frac{dx}{x^2 \sqrt{X}} = -\frac{\sqrt{X}}{a^2 x}$$

$$170. \int \frac{dx}{x^3 \sqrt{X}} = -\frac{\sqrt{X}}{2a^2 x^2} - \frac{1}{2a^3} \log_e \frac{a + \sqrt{X}}{x}$$

$$171. \int \sqrt{X^3} dx = \frac{1}{4} \left(x \sqrt{X^3} + \frac{3a^2 x}{2} \sqrt{X} + \frac{3a^4}{2} \arcsin \frac{x}{a} \right)$$

$$172. \int x \sqrt{X^3} dx = -\frac{1}{5} \sqrt{X^5}$$

$$173. \int x^2 \sqrt{X^3} dx = -\frac{x \sqrt{X^5}}{6} + \frac{a^2 x \sqrt{X^3}}{24} + \frac{a^4 x \sqrt{X}}{16} + \frac{a^6}{16} \arcsin \frac{x}{a}$$

$$174. \int x^3 \sqrt{X^3} dx = \frac{\sqrt{X^7}}{7} - \frac{a^2 \sqrt{X^5}}{5}$$

$$175. \int \frac{\sqrt{X^3}}{x} dx = \frac{\sqrt{X^3}}{3} + a^2 \sqrt{X} - a^3 \log_e \frac{a + \sqrt{X}}{x}$$

$$176. \int \frac{\sqrt{X^3}}{x^2} dx = -\frac{\sqrt{X^3}}{x} - \frac{3}{2} x \sqrt{X} - \frac{3}{2} a^2 \arcsin \frac{x}{a}$$

$$177. \int \frac{\sqrt{X^3}}{x^3} dx = -\frac{\sqrt{X^3}}{2x^2} - \frac{3\sqrt{X}}{2} + \frac{3a}{2} \log_e \frac{a + \sqrt{X}}{x}$$

$$178. \int \frac{dx}{\sqrt{X^3}} = \frac{x}{a^2 \sqrt{X}}$$

$$179. \int \frac{x dx}{\sqrt{X^3}} = \frac{1}{\sqrt{X}}$$

$$180. \int \frac{x^2 dx}{\sqrt{X^3}} = \frac{x}{\sqrt{X}} - \arcsin \frac{x}{a}$$

$$181. \int \frac{x^3 dx}{\sqrt{X^3}} = \sqrt{X} + \frac{a^2}{\sqrt{X}}$$

$$182. \int \frac{dx}{x \sqrt{X^3}} = \frac{1}{a^2 \sqrt{X}} - \frac{1}{a^3} \log_e \frac{a + \sqrt{X}}{x}$$

$$183. \int \frac{dx}{x^2 \sqrt{X^3}} = \frac{1}{a^4} \left(-\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}} \right)$$

$$184. \int \frac{dx}{x^3 \sqrt{X^3}} = -\frac{1}{2a^2 x^2 \sqrt{X}} + \frac{3}{2a^4 \sqrt{X}} - \frac{3}{2a^5} \log_e \frac{a + \sqrt{X}}{x}$$

(m) Integrals containing $\sqrt{x^2 + a^2}$, with

$$X \equiv x^2 + a^2 \quad (a > 0)$$

$$185. \int \sqrt{X} dx = \frac{1}{2} \left(x \sqrt{X} + a^2 \sinh^{-1} \frac{x}{a} \right)$$

$$= \frac{1}{2} [x \sqrt{X} + a^2 \log_e (x + \sqrt{X})] + C_1$$

186. $\int x\sqrt{X} dx = \frac{1}{3}\sqrt{X^3}$
187. $\int x^2\sqrt{X} dx = \frac{x}{4}\sqrt{X^3} - \frac{a^2}{8}\left(x\sqrt{X} + a^2\sinh^{-1}\frac{x}{a}\right)$
 $= \frac{x}{4}\sqrt{X^3} - \frac{a^2}{8}[x\sqrt{X} + a^2\log_e(x + \sqrt{X})] + C_1$
188. $\int x^3\sqrt{X} dx = \frac{\sqrt{X^5}}{5} - \frac{a^2\sqrt{X^3}}{3}$
189. $\int \frac{\sqrt{X}}{x} dx = \sqrt{X} - a\log_e\frac{a + \sqrt{X}}{x}$
190. $\int \frac{\sqrt{X}}{x^2} dx = -\frac{\sqrt{X}}{x} + \sinh^{-1}\frac{x}{a}$
 $= -\frac{\sqrt{X}}{x} + \log_e(x + \sqrt{X}) + C_1$
191. $\int \frac{\sqrt{X}}{x^3} dx = -\frac{\sqrt{X}}{2x^2} - \frac{1}{2a}\log_e\frac{a + \sqrt{X}}{x}$
192. $\int \frac{dx}{\sqrt{X}} = \sinh^{-1}\frac{x}{a} = \log_e(x + \sqrt{X}) + C_1$
193. $\int \frac{x dx}{\sqrt{X}} = \sqrt{X}$
194. $\int \frac{x^2 dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} - \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$
 $= \frac{x}{2}\sqrt{X} - \frac{a^2}{2}\log_e(x + \sqrt{X}) + C_1$
195. $\int \frac{x^3 dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} - a^2\sqrt{X}$
196. $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\log_e\frac{a + \sqrt{X}}{x}$
197. $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x}$

$$198. \int \frac{dx}{x^3 \sqrt{X}} = -\frac{\sqrt{X}}{2a^2 x^2} + \frac{1}{2a^3} \log_e \frac{a + \sqrt{X}}{x}$$

$$199. \int \sqrt{X^3} dx = \frac{1}{4} \left(x\sqrt{X^3} + \frac{3a^2 x}{2} \sqrt{X} + \frac{3a^4}{2} \sinh^{-1} \frac{x}{a} \right) \\ = \frac{1}{4} \left[x\sqrt{X^3} + \frac{3a^2 x}{2} \sqrt{X} + \frac{3a^4}{2} \log_e (x + \sqrt{X}) \right] + C_1$$

$$200. \int x\sqrt{X^3} dx = \frac{1}{5} \sqrt{X^5}$$

$$201. \int x^2 \sqrt{X^3} dx = \frac{x\sqrt{X^6}}{6} - \frac{a^2 x \sqrt{X^3}}{24} - \frac{a^4 x \sqrt{X}}{16} - \frac{a^6}{16} \sinh^{-1} \frac{x}{a} \\ = \frac{x\sqrt{X^6}}{6} - \frac{a^2 x \sqrt{X^3}}{24} - \frac{a^4 x \sqrt{X}}{16} \\ - \frac{a^6}{16} \log_e (x + \sqrt{X}) + C_1$$

$$202. \int x^3 \sqrt{X^3} dx = \frac{\sqrt{X^7}}{7} - \frac{a^2 \sqrt{X^5}}{5}$$

$$203. \int \frac{\sqrt{X^3}}{x} dx = \frac{\sqrt{X^3}}{3} + a^2 \sqrt{X} - a^3 \log_e \frac{a + \sqrt{X}}{x}$$

$$204. \int \frac{\sqrt{X^3}}{x^2} dx = -\frac{\sqrt{X^3}}{x} + \frac{3}{2} x \sqrt{X} + \frac{3}{2} a^2 \sinh^{-1} \frac{x}{a} \\ = -\frac{\sqrt{X^3}}{x} + \frac{3}{2} x \sqrt{X} + \frac{3}{2} a^2 \log_e (x + \sqrt{X}) + C_1$$

$$205. \int \frac{\sqrt{X^3}}{x^3} dx = -\frac{\sqrt{X^3}}{2x^2} + \frac{3}{2} \sqrt{X} - \frac{3}{2} a \log_e \left(\frac{a + \sqrt{X}}{x} \right)$$

$$206. \int \frac{dx}{\sqrt{X^3}} = \frac{x}{a^2 \sqrt{X}}$$

$$207. \int \frac{x dx}{\sqrt{X^3}} = -\frac{1}{\sqrt{X}}$$

$$208. \int \frac{x^2 dx}{\sqrt{X^3}} = -\frac{x}{\sqrt{X}} + \sinh^{-1} \frac{x}{a} \\ = -\frac{x}{\sqrt{X}} + \log_e (x + \sqrt{X}) + C_1$$

$$209. \int \frac{x^2 dx}{\sqrt{X^3}} = \sqrt{X} + \frac{a^2}{\sqrt{X}}$$

$$210. \int \frac{dx}{x\sqrt{X^3}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3} \log_e \frac{a + \sqrt{X}}{x}$$

$$211. \int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{a^4} \left(\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}} \right)$$

$$212. \int \frac{dx}{x^3\sqrt{X^3}} = -\frac{1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} + \frac{3}{2a^6} \log_e \frac{a + \sqrt{X}}{x}$$

(n) Integrals containing $\sqrt{x^2 - a^2}$, with

$$X \equiv x^2 - a^2 \quad (a > 0)$$

$$213. \int \sqrt{X} dx = \frac{1}{2} \left(x\sqrt{X} - a^2 \cosh^{-1} \frac{x}{a} \right)$$

$$= \frac{1}{2} [x\sqrt{X} - a^2 \log_e (x + \sqrt{X})] + C_1$$

$$214. \int x\sqrt{X} dx = \frac{1}{3}\sqrt{X^3}$$

$$215. \int x^2\sqrt{X} dx = \frac{x}{4}\sqrt{X^3} + \frac{a^2}{8} \left(x\sqrt{X} - a^2 \cosh^{-1} \frac{x}{a} \right)$$

$$= \frac{x}{4}\sqrt{X^3} + \frac{a^2}{8} [x\sqrt{X} - a^2 \log_e (x + \sqrt{X})] + C_1$$

$$216. \int x^3\sqrt{X} dx = \frac{\sqrt{X^5}}{5} + \frac{a^2\sqrt{X^3}}{3}$$

$$217. \int \frac{\sqrt{X}}{x} dx = \sqrt{X} - a \arccos \frac{a}{x}$$

$$218. \int \frac{\sqrt{X}}{x^2} dx = -\frac{\sqrt{X}}{x} + \cosh^{-1} \frac{x}{a}$$

$$= -\frac{\sqrt{X}}{x} + \log_e (x + \sqrt{X}) + C_1$$

$$219. \int \frac{\sqrt{X}}{x^3} dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a} \arccos \frac{a}{x}$$

$$220. \int \frac{dx}{\sqrt{X}} = \cosh^{-1} \frac{x}{a} = \log_e (x + \sqrt{X}) + C_1$$

$$221. \int \frac{x dx}{\sqrt{X}} = \sqrt{X}$$

$$222. \int \frac{x^2 dx}{\sqrt{X}} = \frac{x}{2} \sqrt{X} + \frac{a^2}{2} \cosh^{-1} \frac{x}{a}$$

$$= \frac{x}{2} \sqrt{X} + \frac{a^2}{2} \log_e (x + \sqrt{X}) + C_1$$

$$223. \int \frac{x^3 dx}{\sqrt{X}} = \frac{\sqrt{X}^3}{3} + a^2 \sqrt{X}$$

$$224. \int \frac{dx}{x\sqrt{X}} = \frac{1}{a} \arccos \frac{a}{x}$$

$$225. \int \frac{dx}{x^2\sqrt{X}} = \frac{\sqrt{X}}{a^2 x}$$

$$226. \int \frac{dx}{x^3\sqrt{X}} = \frac{\sqrt{X}}{2a^2 x^2} + \frac{1}{2a^3} \arccos \frac{a}{x}$$

$$227. \int \sqrt{X}^3 dx = \frac{1}{4} \left(x\sqrt{X}^3 - \frac{3a^2 x}{2} \sqrt{X} + \frac{3a^4}{2} \cosh^{-1} \frac{x}{a} \right) \\ = \frac{1}{4} \left[x\sqrt{X}^3 - \frac{3a^2 x}{2} \sqrt{X} + \frac{3a^4}{2} \log_e (x + \sqrt{X}) \right] + C_1$$

$$228. \int x\sqrt{X}^3 dx = \frac{1}{5} \sqrt{X}^5$$

$$229. \int x^2\sqrt{X}^3 dx = \frac{x\sqrt{X}^5}{6} + \frac{a^2 x\sqrt{X}^3}{24} - \frac{a^4 x\sqrt{X}}{16} + \frac{a^6}{16} \cosh^{-1} \frac{x}{a} \\ = \frac{x\sqrt{X}^5}{6} + \frac{a^2 x\sqrt{X}^3}{24} - \frac{a^4 x\sqrt{X}}{16} \\ + \frac{a^6}{16} \log_e (x + \sqrt{X}) + C_1$$

$$230. \int x^3 \sqrt{X^3} dx = \frac{\sqrt{X^7}}{7} + \frac{a^2 \sqrt{X^5}}{5}$$

$$231. \int \frac{\sqrt{X^3}}{x} dx = \frac{\sqrt{X^3}}{3} - a^2 \sqrt{X} + a^3 \arccos \frac{a}{x}$$

$$232. \int \frac{\sqrt{X^3}}{x^2} dx = -\frac{\sqrt{X^3}}{2} + \frac{3}{2} x \sqrt{X} - \frac{3}{2} a^2 \cosh^{-1} \frac{x}{a}$$

$$= -\frac{\sqrt{X^3}}{2} + \frac{3}{2} x \sqrt{X} - \frac{3}{2} a^2 \log_e (x + \sqrt{X}) + C_1$$

$$233. \int \frac{\sqrt{X^3}}{x^3} dx = -\frac{\sqrt{X^3}}{2x^2} + \frac{3\sqrt{X}}{2} - \frac{3}{2} a \arccos \frac{a}{x}$$

$$234. \int \frac{dx}{\sqrt{X^3}} = -\frac{x}{a^2 \sqrt{X}}$$

$$235. \int \frac{x dx}{\sqrt{X^3}} = -\frac{1}{\sqrt{X}}$$

$$236. \int \frac{x^2 dx}{\sqrt{X^3}} = -\frac{x}{\sqrt{X}} + \cosh^{-1} \frac{x}{a}$$

$$= -\frac{x}{\sqrt{X}} + \log_e (x + \sqrt{X}) + C_1$$

$$237. \int \frac{x^3 dx}{\sqrt{X^3}} = \sqrt{X} - \frac{a^2}{\sqrt{X}}$$

$$238. \int \frac{dx}{x \sqrt{X^3}} = -\frac{1}{a^2 \sqrt{X}} - \frac{1}{a^3} \arccos \frac{a}{x}$$

$$239. \int \frac{dx}{x^2 \sqrt{X^3}} = -\frac{1}{a^4} \left(\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}} \right)$$

$$240. \int \frac{dx}{x^3 \sqrt{X^3}} = \frac{1}{2a^2 x^2 \sqrt{X}} - \frac{3}{2a^4 \sqrt{X}} - \frac{3}{2a^5} \arccos \frac{a}{x}$$

(o) Integrals containing $\sqrt{ax^2 + bx + c}$, with

$$X \equiv ax^2 + bx + c$$

$$\Delta = 4ac - b^2 \quad k = \frac{4a}{4ac - b^2}$$

$$(a \neq 0)$$

$$241. \int \frac{dx}{\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{a}} \log_e (2\sqrt{aX} + 2ax + b) + C & (a > 0) \\ \frac{1}{\sqrt{a}} \sinh^{-1} \frac{2ax + b}{\sqrt{a}} + C_1 & (a > 0, \Delta > 0) \\ \frac{1}{\sqrt{a}} \log_e (2ax + b) & (a > 0, \Delta = 0) \\ -\frac{1}{\sqrt{-a}} \arcsin \frac{2ax + b}{\sqrt{-\Delta}} & (a < 0, \Delta < 0) \end{cases}$$

$$242. \int \frac{dx}{X\sqrt{X}} = \frac{2(2ax + b)}{\Delta\sqrt{X}} \quad (\Delta \neq 0)$$

$$243. \int \frac{dx}{X^2\sqrt{X}} = \frac{2(2ax + b)}{3\Delta\sqrt{X}} \left(\frac{1}{X} + 2k \right) \quad (\Delta \neq 0)$$

$$244. \int \frac{dx}{X^{(2n+1)/2}} = \frac{2(2ax + b)}{(2n-1)\Delta X^{(2n-1)/2}} + \frac{2k(n-1)}{2n-1} \int \frac{dx}{X^{(2n-1)/2}}$$

($\Delta \neq 0$)

$$245. \int \sqrt{X} dx = \frac{(2ax + b)\sqrt{X}}{4a} + \frac{1}{2k} \int \frac{dx}{\sqrt{X}}$$

$$246. \int X\sqrt{X} dx = \frac{(2ax + b)\sqrt{X}}{8a} \left(X + \frac{3}{2k} \right) + \frac{3}{8k^2} \int \frac{dx}{\sqrt{X}}$$

$$247. \int X^2\sqrt{X} dx = \frac{(2ax + b)\sqrt{X}}{12a} \left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2} \right) + \frac{5}{16k^3} \int \frac{dx}{\sqrt{X}}$$

$$248. \int X^{(2n+1)/2} dx = \frac{(2ax + b)X^{(2n+1)/2}}{4a(n+1)} + \frac{2n+1}{2k(n+1)} \int X^{(2n-1)/2} dx$$

$$249. \int \frac{x dx}{\sqrt{X}} = \frac{\sqrt{X}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{X}}$$

$$250. \int \frac{x dx}{X\sqrt{X}} = -\frac{2(bx + 2c)}{\Delta\sqrt{X}} \quad (\Delta \neq 0)$$

$$251. \int \frac{x dx}{X^{(2n+1)/2}} = -\frac{1}{(2n-1)aX^{(2n-1)/2}} - \frac{b}{2a} \int \frac{dx}{X^{(2n+1)/2}}$$

$$252. \int \frac{x^2 dx}{\sqrt{X}} = \left(\frac{x}{2a} - \frac{3b}{4a^2} \right) \sqrt{X} + \frac{3b^2 - 4ac}{8a^2} \int \frac{dx}{\sqrt{X}}$$

$$253. \int \frac{x^2 dx}{X\sqrt{X}} = \frac{(2b^2 - 4ac)x + 2bc}{a\Delta\sqrt{X}} + \frac{1}{a} \int \frac{dx}{\sqrt{X}} \quad (\Delta \neq 0)$$

$$254. \int x\sqrt{X} dx = \frac{X\sqrt{X}}{3a} - \frac{b(2ax+b)}{8a^2} \sqrt{X} - \frac{b}{4ak} \int \frac{dx}{\sqrt{X}}$$

$$255. \int xX\sqrt{X} dx = \frac{X^2\sqrt{X}}{5a} - \frac{b}{2a} \int X\sqrt{X} dx$$

$$256. \int xX^{(2n+1)/2} dx = \frac{X^{(2n+3)/2}}{(2n+3)a} - \frac{b}{2a} \int X^{(2n+1)/2} dx$$

$$257. \int x^2\sqrt{X} dx = \left(x - \frac{5b}{6a} \right) \frac{X\sqrt{X}}{4a} + \frac{5b^2 - 4ac}{16a^2} \int \sqrt{X} dx$$

$$258. \int \frac{dx}{x\sqrt{X}} = \begin{cases} -\frac{1}{\sqrt{c}} \log_e \left(\frac{2\sqrt{cX}}{x} + \frac{2c}{x} + b \right) + C & (c > 0) \\ -\frac{1}{\sqrt{c}} \sinh^{-1} \frac{bx + 2c}{x\sqrt{\Delta}} + C_1 & (c > 0, \Delta > 0) \\ -\frac{1}{\sqrt{c}} \log_e \frac{bx + 2c}{x} & (c > 0, \Delta = 0) \\ \frac{1}{\sqrt{-c}} \arcsin \frac{bx + 2c}{x\sqrt{-\Delta}} & (c < 0, \Delta < 0) \end{cases}$$

$$259. \int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{cx} - \frac{b}{2c} \int \frac{dx}{x\sqrt{X}}$$

$$260. \int \frac{\sqrt{X} dx}{x} = \sqrt{X} + \frac{b}{2} \int \frac{dx}{\sqrt{X}} + c \int \frac{dx}{x\sqrt{X}}$$

$$261. \int \frac{\sqrt{X} dx}{x^2} = -\frac{\sqrt{X}}{x} + a \int \frac{dx}{\sqrt{X}} + \frac{b}{2} \int \frac{dx}{x\sqrt{X}}$$

$$262. \int \frac{X^{(2n+1)/2}}{x} dx = \frac{X^{(2n+1)/2}}{2n+1} + \frac{b}{2} \int X^{(2n-1)/2} dx + c \int \frac{X^{(2n-1)/2}}{x} dx$$

(p) Other irrational forms ($a > 0, b \neq 0$)

$$263. \int \frac{dx}{x\sqrt{ax^2+bx}} = -\frac{2}{bx} \sqrt{ax^2+bx}$$

$$264. \int \frac{dx}{\sqrt{2ax-x^2}} = \arcsin \frac{x-a}{a}$$

$$265. \int \frac{x dx}{\sqrt{2ax-x^2}} = -\sqrt{2ax-x^2} + a \arcsin \frac{x-a}{a}$$

$$266. \int \sqrt{2ax-x^2} dx = \frac{x-a}{2} \sqrt{2ax-x^2} + \frac{a^2}{2} \arcsin \frac{x-a}{a}$$

$$267. \int \frac{dx}{(ax^2+b)\sqrt{fx^2+g}} = \frac{1}{\sqrt{b}\sqrt{ag-bf}} \arctan \frac{x\sqrt{ag-bf}}{\sqrt{b}\sqrt{fx^2+g}}$$

($ag-bf > 0$)

$$= \frac{1}{2\sqrt{b}\sqrt{bf-ag}} \log_e \frac{\sqrt{b}\sqrt{fx^2+g} + x\sqrt{bf-ag}}{\sqrt{b}\sqrt{fx^2+g} - x\sqrt{bf-ag}} \quad (ag-bf < 0)$$

$$268. \int \sqrt[n]{ax+b} dx = \frac{n(ax+b)^{\frac{n}{n+1}} \sqrt[n]{ax+b}}{(n+1)a}$$

$$269. \int \frac{dx}{\sqrt[n]{ax+b}} = \frac{n(ax+b)^{\frac{1}{n}}}{(n-1)a \sqrt[n]{ax+b}}$$

$$270. \int \frac{dx}{x\sqrt{x^n+a^2}} = -\frac{2}{na} \log_e \frac{a + \sqrt{x^n+a^2}}{\sqrt{x^n}}$$

$$271. \int \frac{dx}{x\sqrt{x^n-a^2}} = \frac{2}{na} \arccos \frac{a}{\sqrt{x^n}}$$

$$272. \int \frac{\sqrt{x} dx}{\sqrt{a^3-x^3}} = \frac{2}{3} \arcsin \sqrt{\left(\frac{x}{a}\right)^3}$$

(q) Recursion formulas (m, n, p are integers)

$$273. \int x^m(ax^n+b)^p dx$$

$$= \frac{1}{m+np+1} \left[x^{m+1}(ax^n+b)^p + npb \int x^m(ax^n+b)^{p-1} dx \right]$$

$$\begin{aligned}
 &= \frac{1}{bn(p+1)} \left[-x^{m+1}(ax^n + b)^{p+1} \right. \\
 &\quad \left. + (m+n+np+1) \int x^m(ax^n + b)^{p+1} dx \right] \\
 &= \frac{1}{(m+1)b} \left[x^{m+1}(ax^n + b)^{p+1} \right. \\
 &\quad \left. - a(m+n+np+1) \int x^{m+n}(ax^n + b)^p dx \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{a(m+np+1)} \left[x^{m-n+1}(ax^n + b)^{p+1} \right. \\
 &\quad \left. - (m-n+1)b \int x^{m-n}(ax^n + b)^p dx \right]
 \end{aligned}$$

(r) *Integrals containing the sine function* ($a \neq 0$)

$$274. \int \sin ax \, dx = -\frac{1}{a} \cos ax$$

$$275. \int \sin^2 ax \, dx = \frac{1}{2}x - \frac{1}{4a} \sin 2ax$$

$$276. \int \sin^3 ax \, dx = -\frac{1}{a} \cos ax + \frac{1}{3a} \cos^3 ax$$

$$277. \int \sin^4 ax \, dx = \frac{3}{8}x - \frac{1}{4a} \sin 2ax + \frac{1}{32a} \sin 4ax$$

$$278. \int \sin^n ax \, dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n} \int \sin^{n-2} ax \, dx \quad (n > 0)$$

$$279. \int x \sin ax \, dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a}$$

$$280. \int x^2 \sin ax \, dx = \frac{2x}{a^2} \sin ax - \left(\frac{x^2}{a} - \frac{2}{a^3} \right) \cos ax$$

$$281. \int x^3 \sin ax \, dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4} \right) \sin ax - \left(\frac{x^3}{a} - \frac{6x}{a^3} \right) \cos ax$$

$$282. \int x^n \sin ax \, dx = -\frac{x^n}{a} \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \, dx$$

$$283. \int \frac{\sin ax}{x} \, dx = ax - \frac{(ax)^3}{3 \cdot 3!} + \frac{(ax)^5}{5 \cdot 5!} - \frac{(ax)^7}{7 \cdot 7!} + \dots$$

284. $\int \frac{\sin ax}{x^2} dx = -\frac{\sin ax}{x} + a \int \frac{\cos ax}{x} dx$
285. $\int \frac{\sin ax}{x^n} dx = -\frac{1}{n-1} \frac{\sin ax}{x^{n-1}} + \frac{a}{n-1} \int \frac{\cos ax}{x^{n-1}} dx$
286. $\int \frac{dx}{\sin ax} = \int \operatorname{cosec} ax \, dx = \frac{1}{a} \log_e \tan \frac{ax}{2}$
 $= \frac{1}{a} \log_e (\operatorname{cosec} ax - \cot ax)$
287. $\int \frac{dx}{\sin^2 ax} = -\frac{1}{a} \cot ax$
288. $\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a \sin^2 ax} + \frac{1}{2a} \log_e \tan \frac{ax}{2}$
289. $\int \frac{dx}{\sin^n ax} = -\frac{1}{a(n-1)} \frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} ax} \quad (n > 1)$
290. $\int \frac{x \, dx}{\sin ax} = \frac{1}{a^2} \left[ax + 3 \cdot 3! + \frac{(ax)^3}{3 \cdot 3!} + \frac{7(ax)^5}{3 \cdot 5 \cdot 5!} + \frac{31(ax)^7}{3 \cdot 7 \cdot 7!} \right.$
 $\left. + \frac{127(ax)^9}{3 \cdot 5 \cdot 9!} + \dots \right]$
291. $\int \frac{x \, dx}{\sin^2 ax} = -\frac{x}{a} \cot ax + \frac{1}{a^2} \log_e \sin ax$
292. $\int \frac{x \, dx}{\sin^n ax} = -\frac{x \cos ax}{(n-1)a \sin^{n-1} ax} - \frac{1}{(n-1)(n-2)a^2 \sin^{n-2} ax}$
 $+ \frac{n-2}{n-1} \int \frac{x \, dx}{\sin^{n-2} ax} \quad (n > 2)$
293. $\int \frac{dx}{1 + \sin ax} = -\frac{1}{a} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right)$
294. $\int \frac{dx}{1 - \sin ax} = \frac{1}{a} \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right)$
295. $\int \frac{x \, dx}{1 + \sin ax} = -\frac{x}{a} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right) + \frac{2}{a^2} \log_e \cos \left(\frac{\pi}{4} - \frac{ax}{2} \right)$
296. $\int \frac{x \, dx}{1 - \sin ax} = \frac{x}{a} \cot \left(\frac{\pi}{4} - \frac{ax}{2} \right) + \frac{2}{a^2} \log_e \sin \left(\frac{\pi}{4} - \frac{ax}{2} \right)$

297. $\int \frac{\sin ax dx}{1 \pm \sin ax} = \pm x + \frac{1}{a} \tan \left(\frac{\pi}{4} \mp \frac{ax}{2} \right)$
298. $\int \frac{dx}{\sin ax (1 \pm \sin ax)} = \frac{1}{a} \tan \left(\frac{\pi}{4} \mp \frac{ax}{2} \right) + \frac{1}{a} \log_e \tan \frac{ax}{2}$
299. $\int \frac{dx}{(1 + \sin ax)^2} = -\frac{1}{2a} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right) - \frac{1}{6a} \tan^3 \left(\frac{\pi}{4} - \frac{ax}{2} \right)$
300. $\int \frac{dx}{(1 - \sin ax)^2} = \frac{1}{2a} \cot \left(\frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \cot^3 \left(\frac{\pi}{4} - \frac{ax}{2} \right)$
301. $\int \frac{\sin ax dx}{(1 + \sin ax)^2} = -\frac{1}{2a} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \tan^3 \left(\frac{\pi}{4} - \frac{ax}{2} \right)$
302. $\int \frac{\sin ax dx}{(1 - \sin ax)^2} = -\frac{1}{2a} \cot \left(\frac{\pi}{4} - \frac{ax}{2} \right) + \frac{1}{6a} \cot^3 \left(\frac{\pi}{4} - \frac{ax}{2} \right)$
303. $\int \frac{dx}{1 + \sin^2 ax} = \frac{1}{2\sqrt{2}a} \arcsin \left(\frac{3 \sin^2 ax - 1}{\sin^2 ax + 1} \right)$
304. $\int \frac{dx}{1 - \sin^2 ax} = \int \frac{dx}{\cos^2 ax} = \frac{1}{a} \tan ax$
305. $\int \sin ax \sin bx dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}$
 $(|a| \neq |b|; \text{ for } |a| = |b| \text{ see 275})$
306. $\int \frac{dx}{b + c \sin ax} = \frac{2}{a\sqrt{b^2 - c^2}} \arctan \frac{b \tan(ax/2) + c}{\sqrt{b^2 - c^2}}$
 $= \frac{1}{a\sqrt{c^2 - b^2}} \log_e \frac{b \tan(ax/2) + c - \sqrt{c^2 - b^2}}{b \tan(ax/2) + c + \sqrt{c^2 - b^2}}$
 $(b^2 < c^2)$
307. $\int \frac{\sin ax dx}{b + c \sin ax} = \frac{x}{c} - \frac{b}{c} \int \frac{dx}{b + c \sin ax}$
308. $\int \frac{dx}{\sin ax (b + c \sin ax)} = \frac{1}{ab} \log_e \tan \frac{ax}{2} - \frac{c}{b} \int \frac{dx}{b + c \sin ax}$
309. $\int \frac{dx}{(b + c \sin ax)^2} = \frac{c \cos ax}{a(b^2 - c^2)(b + c \sin ax)}$
 $+ \frac{b}{b^2 - c^2} \int \frac{dx}{b + c \sin ax}$

$$310. \int \frac{\sin ax \, dx}{(b + c \sin ax)^2} = \frac{b \cos ax}{a(c^2 - b^2)(b + c \sin ax)} + \frac{c}{c^2 - b^2} \int \frac{dx}{b + c \sin ax}$$

$$311. \int \frac{dx}{b^2 + c^2 \sin^2 ax} = \frac{1}{ab\sqrt{b^2 + c^2}} \arctan \frac{\sqrt{b^2 + c^2} \tan ax}{b} \quad (b > 0)$$

$$312. \int \frac{dx}{b^2 - c^2 \sin^2 ax} = \frac{1}{ab\sqrt{b^2 - c^2}} \arctan \frac{\sqrt{b^2 - c^2} \tan ax}{b} \quad (b^2 > c^2, b > 0)$$

$$= \frac{1}{2ab\sqrt{c^2 - b^2}} \log_e \frac{\sqrt{c^2 - b^2} \tan ax + b}{\sqrt{c^2 - b^2} \tan ax - b} \quad (c^2 > b^2, b > 0)$$

(s) *Integrals containing the cosine function* ($a \neq 0$)

$$313. \int \cos ax \, dx = \frac{1}{a} \sin ax$$

$$314. \int \cos^2 ax \, dx = \frac{1}{2}x + \frac{1}{4a} \sin 2ax$$

$$315. \int \cos^3 ax \, dx = \frac{1}{a} \sin ax - \frac{1}{3a} \sin^3 ax$$

$$316. \int \cos^4 ax \, dx = \frac{3}{8}x + \frac{1}{4a} \sin 2ax + \frac{1}{32a} \sin 4ax$$

$$317. \int \cos^n ax \, dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n} \int \cos^{n-2} ax \, dx$$

$$318. \int x \cos ax \, dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a}$$

$$319. \int x^2 \cos ax \, dx = \frac{2x}{a^2} \cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3} \right) \sin ax$$

$$320. \int x^3 \cos ax \, dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4} \right) \cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3} \right) \sin ax$$

$$321. \int x^n \cos ax \, dx = \frac{x^n \sin ax}{a} - \frac{n}{a} \int x^{n-1} \sin ax \, dx \quad (n > 0)$$

$$322. \int \frac{\cos ax}{x} dx = \log_e(ax) - \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^4}{4 \cdot 4!} - \frac{(ax)^6}{6 \cdot 6!} + \dots$$

$$323. \int \frac{\cos ax}{x^2} dx = -\frac{\cos ax}{x} - a \int \frac{\sin ax}{x} dx$$

$$324. \int \frac{\cos ax}{x^n} dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1} \int \frac{\sin ax}{x^{n-1}} dx \quad (n \neq 1)$$

$$325. \int \frac{dx}{\cos ax} = \frac{1}{a} \log_e \tan \left(\frac{ax}{2} + \frac{\pi}{4} \right) = \frac{1}{a} \log_e (\sec ax + \tan ax)$$

$$326. \int \frac{dx}{\cos^2 ax} = \frac{1}{a} \tan ax$$

$$327. \int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a \cos^2 ax} + \frac{1}{2a} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right)$$

$$328. \int \frac{dx}{\cos^n ax} = \frac{1}{a(n-1)} \frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} ax} \quad (n > 1)$$

$$329. \int \frac{x dx}{\cos ax} = \frac{1}{a^2} \left[\frac{(ax)^2}{2} + 4 \cdot 2! + \frac{(ax)^4}{6 \cdot 4!} + \frac{5(ax)^6}{8 \cdot 6!} \right.$$

$$\left. + \frac{1,385(ax)^{10}}{10 \cdot 8!} + \dots \right]$$

$$330. \int \frac{x dx}{\cos^2 ax} = \frac{x}{a} \tan ax + \frac{1}{a^2} \log_e \cos ax$$

$$331. \int \frac{x dx}{\cos^n ax} = \frac{x \sin ax}{(n-1)a \cos^{n-1} ax} - \frac{1}{(n-1)(n-2)a^2 \cos^{n-2} ax}$$

$$+ \frac{n-2}{n-1} \int \frac{x dx}{\cos^{n-2} ax} \quad (n > 2)$$

$$332. \int \frac{dx}{1 + \cos ax} = \frac{1}{a} \tan \frac{ax}{2}$$

$$333. \int \frac{dx}{1 - \cos ax} = -\frac{1}{a} \cot \frac{ax}{2}$$

$$334. \int \frac{x dx}{1 + \cos ax} = \frac{x}{a} \tan \frac{ax}{2} + \frac{2}{a^2} \log_e \cos \frac{ax}{2}$$

$$335. \int \frac{x dx}{1 - \cos ax} = -\frac{x}{a} \cot \frac{ax}{2} + \frac{2}{a^2} \log_e \sin \frac{ax}{2}$$

336. $\int \frac{\cos ax \, dx}{1 + \cos ax} = x - \frac{1}{a} \tan \frac{ax}{2}$
337. $\int \frac{\cos ax \, dx}{1 - \cos ax} = -x - \frac{1}{a} \cot \frac{ax}{2}$
338. $\int \frac{dx}{\cos ax (1 + \cos ax)} = \frac{1}{a} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{a} \tan \frac{ax}{2}$
339. $\int \frac{dx}{\cos ax (1 - \cos ax)} = \frac{1}{a} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{a} \cot \frac{ax}{2}$
340. $\int \frac{dx}{(1 + \cos ax)^2} = \frac{1}{2a} \tan \frac{ax}{2} + \frac{1}{6a} \tan^3 \frac{ax}{2}$
341. $\int \frac{dx}{(1 - \cos ax)^2} = -\frac{1}{2a} \cot \frac{ax}{2} - \frac{1}{6a} \cot^3 \frac{ax}{2}$
342. $\int \frac{\cos ax \, dx}{(1 + \cos ax)^2} = \frac{1}{2a} \tan \frac{ax}{2} - \frac{1}{6a} \tan^3 \frac{ax}{2}$
343. $\int \frac{\cos ax \, dx}{(1 - \cos ax)^2} = \frac{1}{2a} \cot \frac{ax}{2} - \frac{1}{6a} \cot^3 \frac{ax}{2}$
344. $\int \frac{dx}{1 + \cos^2 ax} = \frac{1}{2\sqrt{2}a} \arcsin \left(\frac{1 - 3 \cos^2 ax}{1 + \cos^2 ax} \right)$
345. $\int \frac{dx}{1 - \cos^2 ax} = \int \frac{dx}{\sin^2 ax} = -\frac{1}{a} \cot ax$
346. $\int \cos ax \cos bx \, dx = \frac{\sin (a-b)x}{2(a-b)} + \frac{\sin (a+b)x}{2(a+b)}$
347. $\int \frac{dx}{b + c \cos ax} = \frac{2}{a\sqrt{b^2 - c^2}} \arctan \frac{(b-c) \tan (ax/2)}{\sqrt{b^2 - c^2}} \quad (b^2 > c^2)$
 $= \frac{1}{a\sqrt{c^2 - b^2}} \log_e \frac{(c-b) \tan (ax/2) + \sqrt{c^2 - b^2}}{(c+b) \tan (ax/2) - \sqrt{c^2 - b^2}} \quad (b^2 < c^2)$
348. $\int \frac{\cos ax \, dx}{b + c \cos ax} = \frac{x}{c} - \frac{b}{c} \int \frac{dx}{b + c \cos ax}$

$$349. \int \frac{dx}{\cos ax (b + c \cos ax)} = \frac{1}{ab} \log_e \tan \left(\frac{ax}{2} + \frac{\pi}{4} \right) - \frac{c}{b} \int \frac{dx}{b + c \cos ax}$$

$$350. \int \frac{dx}{(b + c \cos ax)^2} = \frac{c \sin ax}{a(c^2 - b^2)(b + c \cos ax)} - \frac{b}{c^2 - b^2} \int \frac{dx}{b + c \cos ax}$$

$$351. \int \frac{\cos ax \, dx}{(b + c \cos ax)^2} = \frac{b \sin ax}{a(b^2 - c^2)(b + c \cos ax)} - \frac{c}{b^2 - c^2} \int \frac{dx}{b + c \cos ax}$$

$$352. \int \frac{dx}{b^2 + c^2 \cos^2 ax} = \frac{1}{ab\sqrt{b^2 + c^2}} \arctan \frac{b \tan ax}{\sqrt{b^2 + c^2}} \quad (b > 0)$$

$$353. \int \frac{dx}{b^2 - c^2 \cos^2 ax} = \frac{1}{ab\sqrt{b^2 - c^2}} \arctan \frac{b \tan ax}{\sqrt{b^2 - c^2}} \quad (b^2 > c^2, b > 0)$$

$$= \frac{1}{2ab\sqrt{c^2 - b^2}} \log_e \frac{b \tan ax - \sqrt{c^2 - b^2}}{b \tan ax + \sqrt{c^2 - b^2}} \quad (c^2 > b^2, b > 0)$$

(t) Integrals containing both sine and cosine $(a \neq 0)$

$$354. \int \sin ax \cos ax \, dx = \frac{1}{2a} \sin^2 ax$$

$$355. \int \sin^2 ax \cos^2 ax \, dx = \frac{x}{8} - \frac{\sin 4ax}{32a}$$

$$356. \int \sin^n ax \cos ax \, dx = \frac{1}{a(n+1)} \sin^{n+1} ax \quad (n \neq -1)$$

$$357. \int \sin ax \cos^n ax \, dx = -\frac{1}{a(n+1)} \cos^{n+1} ax \quad (n \neq -1)$$

$$358. \int \sin^n ax \cos^m ax \, dx$$

$$= -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(n+m)} + \frac{n-1}{n+m} \int \sin^{n-2} ax \cos^m ax \, dx \quad (m, n > 0)$$

$$= \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(n+m)} + \frac{m-1}{n+m} \int \sin^n ax \cos^{m-2} ax \, dx$$

($m, n > 0$)

359.
$$\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a} \log_e \tan ax$$

360.
$$\int \frac{dx}{\sin^2 ax \cos ax} = \frac{1}{a} \left[\log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{\sin ax} \right]$$

361.
$$\int \frac{dx}{\sin ax \cos^2 ax} = \frac{1}{a} \left(\log_e \tan \frac{ax}{2} + \frac{1}{\cos ax} \right)$$

362.
$$\int \frac{dx}{\sin^3 ax \cos ax} = \frac{1}{a} \left(\log_e \tan ax - \frac{1}{2 \sin^2 ax} \right)$$

363.
$$\int \frac{dx}{\sin ax \cos^3 ax} = \frac{1}{a} \left(\log_e \tan ax + \frac{1}{2 \cos^2 ax} \right)$$

364.
$$\int \frac{dx}{\sin^2 ax \cos^2 ax} = -\frac{2}{a} \cot 2ax$$

365.
$$\int \frac{dx}{\sin^2 ax \cos^3 ax} = \frac{1}{a} \left[\frac{\sin ax}{2 \cos^2 ax} - \frac{1}{\sin ax} + \frac{3}{2} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) \right]$$

366.
$$\int \frac{dx}{\sin^3 ax \cos^2 ax} = \frac{1}{a} \left(\frac{1}{\cos ax} - \frac{\cos ax}{2 \sin^2 ax} + \frac{3}{2} \log_e \tan \frac{ax}{2} \right)$$

367.
$$\int \frac{dx}{\sin ax \cos^n ax} = \frac{1}{a(n-1) \cos^{n-1} ax} + \int \frac{dx}{\sin ax \cos^{n-2} ax}$$

($n \neq 1$); (see 361, 363)

368.
$$\int \frac{dx}{\sin^n ax \cos ax} = -\frac{1}{a(n-1) \sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax \cos ax}$$

($n \neq 1$); (see 360, 362)

369.
$$\int \frac{dx}{\sin^n ax \cos^m ax}$$

$$= -\frac{1}{a(n-1)} \cdot \frac{1}{\sin^{n-1} ax \cos^{m-1} ax} + \frac{n+m-2}{n-1} \int \frac{dx}{\sin^{n-2} ax \cos^m ax}$$

($m > 0, n > 1$)

$$= \frac{1}{a(n-1)} \cdot \frac{1}{\sin^{n-1} ax \cos^{m-1} ax} + \frac{n+m-2}{m-1} \int \frac{dx}{\sin^n ax \cos^{m-2} ax}$$

($n > 0, m > 1$)

$$370. \int \frac{\sin ax \, dx}{\cos^2 ax} = \frac{1}{a \cos ax}$$

$$371. \int \frac{\sin ax \, dx}{\cos^3 ax} = \frac{1}{2a \cos^2 ax} = \frac{1}{2a} \tan^2 ax + C_1$$

$$372. \int \frac{\sin ax \, dx}{\cos^n ax} = \frac{1}{a(n-1) \cos^{n-1} ax}$$

$$373. \int \frac{\sin^2 ax \, dx}{\cos ax} = -\frac{1}{a} \sin ax + \frac{1}{a} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right)$$

$$374. \int \frac{\sin^2 ax \, dx}{\cos^3 ax} = \frac{1}{a} \left[\frac{\sin ax}{2 \cos^2 ax} - \frac{1}{2} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) \right]$$

$$375. \int \frac{\sin^2 ax \, dx}{\cos^n ax} = \frac{\sin ax}{a(n-1) \cos^{n-1} ax} - \frac{1}{n-1} \int \frac{dx}{\cos^{n-2} ax} \quad (n \neq 1) \quad (\text{see 325, 326, 328})$$

$$376. \int \frac{\sin^3 ax \, dx}{\cos ax} = -\frac{1}{a} \left(\frac{\sin^2 ax}{2} + \log_e \cos ax \right)$$

$$377. \int \frac{\sin^3 ax \, dx}{\cos^2 ax} = \frac{1}{a} \left(\cos ax + \frac{1}{\cos ax} \right)$$

$$378. \int \frac{\sin^3 ax \, dx}{\cos^n ax} = \frac{1}{a} \left[\frac{1}{(n-1) \cos^{n-1} ax} - \frac{1}{(n-3) \cos^{n-3} ax} \right] \quad (n \neq 1, n \neq 3)$$

$$379. \int \frac{\sin^2 ax \, dx}{\cos ax} = -\frac{\sin^{n-1} ax}{a(n-1)} + \int \frac{\sin^{n-2} ax \, dx}{\cos ax} \quad (n \neq 1)$$

$$380. \int \frac{\sin^m ax \, dx}{\cos^m ax} = \frac{\sin^{n+1} ax}{a(m-1) \cos^{m-1} ax} - \frac{n-m+2}{m-1} \int \frac{\sin^2 ax \, dx}{\cos^{m-2} ax} \quad (m \neq 1)$$

$$= -\frac{\sin^{n-1} ax}{a(n-m) \cos^{m-1} ax} + \frac{n-1}{n-m} \int \frac{\sin^{n-2} ax \, dx}{\cos^m ax} \quad (m \neq n)$$

$$= \frac{\sin^{n-1} ax}{a(m-1) \cos^{m-1} ax} - \frac{n-1}{m-1} \int \frac{\sin^{n-1} ax \, dx}{\cos^{m-2} ax} \quad (m \neq 1)$$

$$381. \int \frac{\cos ax \, dx}{\sin^2 ax} = -\frac{1}{a \sin ax}$$

$$382. \int \frac{\cos ax \, dx}{\sin^3 ax} = -\frac{1}{2a \sin^2 ax}$$

$$383. \int \frac{\cos ax \, dx}{\sin^n ax} = -\frac{1}{a(n-1) \sin^{n-1} ax} \quad (n \neq 1)$$

$$384. \int \frac{\cos^2 ax \, dx}{\sin ax} = \frac{1}{a} \left(\cos ax + \log_e \tan \frac{ax}{2} \right)$$

$$385. \int \frac{\cos^2 ax \, dx}{\sin^3 ax} = -\frac{1}{2a} \left(\frac{\cos ax}{\sin^2 ax} - \log_e \tan \frac{ax}{2} \right)$$

$$386. \int \frac{\cos^2 ax \, dx}{\sin^n ax} = -\frac{1}{n-1} \left(\frac{\cos ax}{\sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax} \right)$$

($n \neq 1$); (see 289)

$$387. \int \frac{\cos^3 ax \, dx}{\sin ax} = \frac{1}{a} \left(\frac{\cos^2 ax}{2} + \log_e \sin ax \right)$$

$$388. \int \frac{\cos^3 ax \, dx}{\sin^2 ax} = -\frac{1}{a} \left(\sin ax + \frac{1}{\sin ax} \right)$$

$$389. \int \frac{\cos^3 ax \, dx}{\sin^n ax} = \frac{1}{a} \left[\frac{1}{(n-3) \sin^{n-3} ax} - \frac{1}{(n-1) \sin^{n-1} ax} \right]$$

($n \neq 1, n \neq 3$)

$$390. \int \frac{\cos^n ax}{\sin ax} \, dx = \frac{\cos^{n-1} ax}{a(n-1)} + \int \frac{\cos^{n-2} ax \, dx}{\sin ax} \quad (n \neq 1)$$

$$391. \int \frac{\cos^n ax \, dx}{\sin^m ax} = -\frac{\cos^{n+1} ax}{a(m-1) \sin^{m-1} ax} - \frac{n-m+2}{m-1} \int \frac{\cos^n ax \, dx}{\sin^{m-2} ax} \quad (m \neq 1)$$

$$= \frac{\cos^{n-1} ax}{a(n-m) \sin^{m-1} ax} + \frac{n-1}{n-m} \int \frac{\cos^{n-2} ax \, dx}{\sin^m ax}$$

($m \neq n$)

$$= -\frac{\cos^{n-1} ax}{a(m-1) \sin^{m-1} ax} - \frac{n-1}{m-1} \int \frac{\cos^{n-2} ax \, dx}{\sin^{m-2} ax}$$

($m \neq 1$)

392. $\int \frac{dx}{\sin ax (1 \pm \cos ax)} = \pm \frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a} \log_e \tan \frac{ax}{2}$
393. $\int \frac{dx}{\cos ax (1 \pm \sin ax)} = \mp \frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right)$
394. $\int \frac{\sin ax \, dx}{\cos ax (1 \pm \cos ax)} = \frac{1}{a} \log_e \frac{1 \pm \cos ax}{\cos ax}$
395. $\int \frac{\cos ax \, dx}{\sin ax (1 \pm \sin ax)} = -\frac{1}{a} \log_e \frac{1 \pm \sin ax}{\sin ax}$
396. $\int \frac{\sin ax \, dx}{\cos ax (1 \pm \sin ax)} = \frac{1}{2a(1 \pm \sin ax)} \pm \frac{1}{2a} \log_e \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right)$
397. $\int \frac{\cos ax \, dx}{\sin ax (1 \pm \cos ax)} = -\frac{1}{2a(1 \pm \cos ax)} \pm \frac{1}{2a} \log_e \tan \frac{ax}{2}$
398. $\int \frac{\sin ax \, dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \frac{1}{2a} \log_e (\sin ax \pm \cos ax)$
399. $\int \frac{\cos ax \, dx}{\sin ax \pm \cos ax} = \pm \frac{x}{2} + \frac{1}{2a} \log_e (\sin ax \pm \cos ax)$
400. $\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}} \log_e \tan \left(\frac{ax}{2} \pm \frac{\pi}{8} \right)$
401. $\int \frac{dx}{1 + \cos ax \pm \sin ax} = \pm \frac{1}{a} \log_e \left(1 \pm \tan \frac{ax}{2} \right)$
402. $\int \frac{dx}{b \sin ax + c \cos ax} = \frac{1}{a\sqrt{b^2 + c^2}} \log_e \tan \frac{ax + \theta}{2}$
 $\sin \theta = \frac{c}{\sqrt{b^2 + c^2}} \quad \tan \theta = \frac{c}{b}$
403. $\int \frac{\sin ax \, dx}{b + c \cos ax} = -\frac{1}{ac} \log_e (b + c \cos ax)$
404. $\int \frac{\cos ax \, dx}{b + c \sin ax} = \frac{1}{ac} \log_e (b + c \sin ax)$
405. $\int \frac{\sin^2 ax \, dx}{b + c \cos^2 ax} = \frac{1}{ac} \sqrt{\frac{b+c}{b}} \arctan \left(\sqrt{\frac{b}{b+c}} \tan ax \right) - \frac{x}{c}$

$$406. \int \frac{\sin ax \cos ax \, dx}{b \cos^2 ax + c \sin^2 ax} = \frac{1}{2a(c-b)} \log_e (b \cos^2 ax + c \sin^2 ax) \quad (c \neq b)$$

$$407. \int \frac{dx}{b^2 \cos^2 ax + c^2 \sin^2 ax} = \frac{1}{abc} \arctan \left(\frac{c}{b} \tan ax \right)$$

$$408. \int \frac{dx}{b^2 \cos^2 ax - c^2 \sin^2 ax} = \frac{1}{2abc} \log_e \frac{c \tan ax + b}{c \tan ax - b}$$

$$409. \int \sin ax \cos bx \, dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)}$$

$(a^2 \neq b^2; \text{ for } a = b \text{ see 354})$

$$410. \int \frac{dx}{b + c \cos ax + d \sin ax}$$

$$\left\{ \frac{-1}{a\sqrt{b^2 - c^2 - d^2}} \arcsin \frac{c^2 + d^2 + b(c \cos ax + d \sin ax)}{\sqrt{c^2 + d^2}(b + c \cos ax + d \sin ax)} \right.$$

$(b^2 > c^2 + d^2, |ax| < \pi)$

$$= \left\{ \frac{1}{a\sqrt{c^2 + d^2 - b^2}} \right. \\ \left. \frac{c^2 + d^2 + b(c \cos ax + d \sin ax)}{\sqrt{c^2 + d^2 - b^2}(c \sin ax - d \cos ax)} + \frac{\sqrt{c^2 + d^2 - b^2}(c \sin ax - d \cos ax)}{\sqrt{c^2 + d^2}(b + c \cos ax + d \sin ax)} \right.$$

$(b^2 < c^2 + d^2, |ax| < \pi)$

$$\left. \frac{1}{ab} \left[\frac{b - (c + d) \cos ax + (c - d) \sin ax}{b + (c - d) \cos ax + (c + d) \sin ax} \right] \right\} \quad (b^2 = c^2 + d^2)$$

(u) Integrals containing tangent and cotangent functions
($a \neq 0$)

$$411. \int \tan ax \, dx = -\frac{1}{a} \log_e \cos ax$$

$$412. \int \tan^2 ax \, dx = \frac{1}{a} \tan ax - x$$

$$413. \int \tan^3 ax \, dx = \frac{1}{2a} \tan^2 ax + \frac{1}{a} \log_e \cos ax$$

$$414. \int \tan^n ax \, dx = \frac{1}{a(n-1)} \tan^{n-1} ax - \int \tan^{n-2} ax \, dx \quad (n > 1)$$

$$415. \int \frac{dx}{b+c \tan ax} = \int \frac{\cot ax \, dx}{b \cot ax + c}$$

$$= \frac{1}{b^2+c^2} \left[bx + \frac{c}{a} \log_e (b \cos ax + c \sin ax) \right]$$

$$416. \int \frac{dx}{\sqrt{b+c \tan^2 ax}} = \frac{1}{a\sqrt{b-c}} \arcsin \left(\sqrt{\frac{b-c}{b}} \sin ax \right)$$

$$(b > 0, b^2 > c^2)$$

$$417. \int \frac{\tan^n ax \, dx}{\cos^2 ax} = \frac{\tan^{n+1} ax}{a(n+1)} \quad (n \neq -1)$$

$$418. \int \cot ax \, dx = \frac{1}{a} \log_e \sin ax$$

$$419. \int \cot^2 ax \, dx = \int \frac{dx}{\tan^2 ax} = -\frac{1}{a} \cot ax - x$$

$$420. \int \cot^3 ax \, dx = -\frac{1}{2a} \cot^2 ax - \frac{1}{a} \log_e \sin ax$$

$$421. \int \cot^n ax \, dx = \int \frac{dx}{\tan^n ax} = -\frac{1}{a(n-1)} \cot^{n-1} ax$$

$$-\int \cot^{n-2} ax \, dx \quad (n > 1)$$

$$422. \int \frac{dx}{b+c \cot ax} = \int \frac{\tan ax \, dx}{b \tan ax + c}$$

$$= \frac{1}{b^2+c^2} \left[bx - \frac{c}{a} \log_e (c \cos ax + b \sin ax) \right]$$

$$423. \int \frac{\cot^n ax \, dx}{\sin^2 ax} = \frac{-\cot^{n+1} ax}{a(n+1)} \quad (n \neq -1)$$

(v) *Integrals containing hyperbolic functions* $(a \neq 0)$

$$424. \int \sinh x \, dx = \cosh x$$

$$425. \int \sinh^2 x \, dx = \frac{\sinh 2x}{4} - \frac{x}{2}$$

$$426. \int \frac{dx}{\sinh x} = \log_e \tanh \left(\frac{x}{2} \right)$$

$$427. \int \frac{dx}{\sinh^2 x} = -\coth x$$

$$428. \int \cosh x \, dx = \sinh x$$

$$429. \int \cosh^2 x \, dx = \frac{\sinh 2x}{4} + \frac{x}{2}$$

$$430. \int \frac{dx}{\cosh x} = 2 \arctan e^x = \arctan (\sinh x)$$

$$431. \int \frac{dx}{\cosh^2 x} = \tanh x$$

$$432. \int \frac{\sinh x}{\cosh^2 x} \, dx = -\frac{1}{\cosh x}$$

$$433. \int \frac{\cosh x}{\sinh^2 x} \, dx = -\frac{1}{\sinh x}$$

$$434. \int x \sinh x \, dx = x \cosh x - \sinh x$$

$$435. \int x \cosh x \, dx = x \sinh x - \cosh x$$

$$436. \int \tanh x \, dx = \log_e \cosh x$$

$$437. \int \tanh^2 x \, dx = x - \tanh x$$

$$438. \int \coth x \, dx = \log_e \sinh x$$

$$439. \int \coth^2 x \, dx = x - \coth x$$

$$440. \int \sinh^n ax \, dx$$

$$= \begin{cases} \frac{1}{an} \sinh^{n-1} ax \cosh ax - \frac{n-1}{n} \int \sinh^{n-2} ax \, dx & (n > 0) \\ \frac{1}{a(n+1)} \sinh^{n+1} ax \cosh ax - \frac{n+2}{n+1} \int \sinh^{n+2} ax \, dx & (n < -1) \end{cases}$$

$$441. \int \cosh^n ax \, dx = \begin{cases} \frac{1}{an} \sinh ax \cosh^{n-1} ax + \frac{n-1}{n} \int \cosh^{n-2} ax \, dx & (n > 0) \\ -\frac{1}{a(n+1)} \sinh ax \cosh^{n+1} ax + \frac{n+2}{n+1} \int \cosh^{n+2} ax \, dx & (n < -1) \end{cases}$$

$$442. \int \sinh ax \sinh bx \, dx = \frac{\sinh(a+b)x}{2(a+b)} - \frac{\sinh(a-b)x}{2(a-b)}$$

$$443. \int \cosh ax \cosh bx \, dx = \frac{\sinh(a+b)x}{2(a+b)} + \frac{\sinh(a-b)x}{2(a-b)} \quad (a^2 \neq b^2)$$

$$444. \int \sinh ax \cosh bx \, dx = \frac{\cosh(a+b)x}{2(a+b)} + \frac{\cosh(a-b)x}{2(a-b)}$$

$$445. \int \sinh ax \sin ax \, dx = \frac{1}{2a} (\cosh ax \sin ax - \sinh ax \cos ax)$$

$$446. \int \cosh ax \cos ax \, dx = \frac{1}{2a} (\sinh ax \cos ax + \cosh ax \sin ax)$$

$$447. \int \sinh ax \cos ax \, dx = \frac{1}{2a} (\cosh ax \cos ax + \sinh ax \sin ax)$$

$$448. \int \cosh ax \sin ax \, dx = \frac{1}{2a} (\sinh ax \sin ax - \cosh ax \cos ax)$$

(w) *Integrals containing exponential functions*

$$449. \int e^{ax} \, dx = \frac{1}{a} e^{ax}$$

$$450. \int x e^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1)$$

$$451. \int x^2 e^{ax} \, dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right)$$

$$452. \int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx \quad (n > 0)$$

$$453. \int \frac{e^{ax}}{x} \, dx = \log_e x + \frac{ax}{1 \cdot 1!} + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^3}{3 \cdot 3!} + \dots$$

$$454. \int \frac{e^{ax}}{x^n} dx = \frac{1}{n-1} \left(-\frac{e^{ax}}{x^{n-1}} + a \int \frac{e^{ax}}{x^{n-1}} dx \right) \quad (n > 1)$$

$$455. \int \frac{dx}{1+e^{ax}} = \frac{1}{a} \log_e \frac{e^{ax}}{1+e^{ax}}$$

$$456. \int \frac{dx}{b+ce^{ax}} = \frac{x}{b} - \frac{1}{ab} \log_e (b+ce^{ax})$$

$$457. \int \frac{e^{ax} dx}{b+ce^{ax}} = \frac{1}{ac} \log_e (b+ce^{ax})$$

$$458. \int \frac{dx}{be^{ax} + ce^{-ax}} = \frac{1}{a\sqrt{bc}} \arctan \left(e^{ax} \sqrt{\frac{b}{c}} \right) \quad (bc > 0)$$

$$= \frac{1}{2a\sqrt{-bc}} \log_e \frac{c + e^{ax}\sqrt{-bc}}{c - e^{ax}\sqrt{-bc}} \quad (bc < 0)$$

$$459. \int \frac{xe^{ax} dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}$$

$$460. \int e^{ax} \log_e x dx = \frac{1}{a} e^{ax} \log_e x - \frac{1}{a} \int \frac{e^{ax}}{x} dx$$

$$461. \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$462. \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

$$463. \int xe^{ax} \sin bx dx = \frac{xe^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$- \frac{e^{ax}}{(a^2 + b^2)^2} [(a^2 - b^2) \sin bx - 2ab \cos bx]$$

$$464. \int xe^{ax} \cos bx dx = \frac{xe^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

$$- \frac{e^{ax}}{(a^2 + b^2)^2} [(a^2 - b^2) \cos bx + 2ab \sin bx]$$

$$465. \int e^{ax} \sin bx \sin cx dx = \frac{e^{ax} [(b-c) \sin (b-c)x + a \cos (b-c)x]}{2[a^2 + (b-c)^2]}$$

$$- \frac{e^{ax} [(b+c) \sin (b+c)x + a \cos (b+c)x]}{2[a^2 + (b+c)^2]}$$

$$466. \int e^{ax} \cos bx \cos cx \, dx = \frac{e^{ax}[(b-c) \sin(b-c)x + a \cos(b-c)x]}{2[a^2 + (b-c)^2]} + \frac{e^{ax}[(b+c) \sin(b+c)x + a \cos(b+c)x]}{2[a^2 + (b+c)^2]}$$

$$467. \int e^{ax} \sin bx \cos cx \, dx = \frac{e^{ax}[a \sin(b-c)x - (b-c) \cos(b-c)x]}{2[a^2 + (b-c)^2]} + \frac{e^{ax}[a \sin(b+c)x - (b+c) \cos(b+c)x]}{2[a^2 + (b+c)^2]}$$

$$468. \int e^{ax} \sin bx \sin (bx+c) \, dx = \frac{e^{ax} \cos c}{2a} - \frac{e^{ax}[a \cos(2bx+c) + 2b \sin(2bx+c)]}{2(a^2 + 4b^2)}$$

$$469. \int e^{ax} \cos bx \cos (bx+c) \, dx = \frac{e^{ax} \cos c}{2a} + \frac{e^{ax}[a \cos(2bx+c) + 2b \sin(2bx+c)]}{2(a^2 + 4b^2)}$$

$$470. \int e^{ax} \sin bx \cos (bx+c) \, dx = -\frac{e^{ax} \sin c}{2a} + \frac{e^{ax}[a \sin(2bx+c) - 2b \cos(2bx+c)]}{2(a^2 + 4b^2)}$$

$$471. \int e^{ax} \cos bx \sin (bx+c) \, dx = \frac{e^{ax} \sin c}{2a} + \frac{e^{ax}[a \sin(2bx+c) - 2b \cos(2bx+c)]}{2(a^2 + 4b^2)}$$

$$472. \int e^{ax} \sin^n bx \, dx = \frac{e^{ax} \sin^{n-1} bx (a \sin bx - n b \cos bx)}{a^2 + n^2 b^2} + \frac{n(n-1)b^2}{a^2 + n^2 b^2} \int e^{ax} \sin^{n-2} bx \, dx$$

$$473. \int e^{ax} \cos^n bx \, dx = \frac{e^{ax} \cos^{n-1} bx (a \cos bx + n b \sin bx)}{a^2 + n^2 b^2} + \frac{n(n-1)b^2}{a^2 + n^2 b^2} \int e^{ax} \cos^{n-2} bx \, dx$$

(x) Integrals containing logarithmic functions (a ≠ 0)

$$474. \int \log_e ax \, dx = x \log_e ax - x$$

$$475. \int (\log_e ax)^2 \, dx = x(\log_e ax)^2 - 2x \log_e ax + 2x$$

$$476. \int (\log_e ax)^n \, dx = x(\log_e ax)^n - n \int (\log_e ax)^{n-1} \, dx \quad (n \neq -1)$$

$$477. \int \frac{dx}{\log_e ax} = \frac{1}{a} \left[\log_e (\log_e ax) + \log_e ax + \frac{(\log_e ax)^2}{2 \cdot 2!} \right.$$

$$\left. + \frac{(\log_e ax)^3}{3 \cdot 3!} + \dots \right]$$

$$478. \int x \log_e ax \, dx = \frac{x^2}{2} \log_e ax - \frac{x^2}{4}$$

$$479. \int x^2 \log_e ax \, dx = \frac{x^3}{3} \log_e ax - \frac{x^3}{9}$$

$$480. \int x^n \log_e ax \, dx = x^{n+1} \left[\frac{\log_e ax}{n+1} - \frac{1}{(n+1)^2} \right] \quad (n \neq -1)$$

$$481. \int x^n (\log_e ax)^m \, dx = \frac{x^{n+1}}{n+1} (\log_e ax)^m - \frac{m}{n+1} \int x^n (\log_e ax)^{m-1} \, dx$$

(m, n ≠ -1)

$$482. \int \frac{(\log_e ax)^n}{x} \, dx = \frac{(\log_e ax)^{n+1}}{n+1} \quad (n \neq -1)$$

$$483. \int \frac{\log_e x}{x^n} \, dx = -\frac{\log_e x}{(n-1)x^{n-1}} - \frac{1}{(n-1)^2 x^{n-1}} \quad (n \neq 1)$$

$$484. \int \frac{(\log_e x)^m}{x^n} \, dx = -\frac{(\log_e x)^m}{(n-1)x^{n-1}} + \frac{m}{n-1} \int \frac{(\log_e x)^{m-1}}{x^n} \, dx$$

(n ≠ 1)

$$485. \int \frac{x^n \, dx}{\log_e ax} = \frac{1}{a^{n+1}} \left[\log_e (\log_e ax) + (n+1) \log_e ax \right.$$
$$\left. + \frac{(n+1)^2 (\log_e ax)^2}{2 \cdot 2!} + \frac{(n+1)^3 (\log_e ax)^3}{3 \cdot 3!} + \dots \right]$$

$$= \frac{1}{a^{n+1}} \int \frac{e^y \, dy}{y} \quad [y = (n+1) \log_e ax]$$

$$486. \int \frac{x^n dx}{(\log_e ax)^m} = \frac{-x^{n+1}}{(m-1)(\log_e ax)^{m-1}} + \frac{n+1}{m-1} \int \frac{x^n dx}{(\log_e ax)^{m-1}} \quad (m \neq 1)$$

$$487. \int \frac{dx}{x \log_e ax} = \log_e (\log_e ax)$$

$$488. \int \frac{dx}{x(\log_e ax)^n} = -\frac{1}{(n-1)(\log_e ax)^{n-1}}$$

$$489. \int \sin (\log_e ax) dx = \frac{x}{2} [\sin (\log_e ax) - \cos (\log_e ax)]$$

$$490. \int \cos (\log_e ax) dx = \frac{x}{2} [\sin (\log_e ax) + \cos (\log_e ax)]$$

$$491. \int e^{ax} \log_e bx dx = \frac{1}{a} e^{ax} \log_e bx - \frac{1}{a} \int \frac{e^{ax}}{x} dx$$

(y) **Integrals containing inverse trigonometric and hyperbolic functions** ($a > 0$)

$$492. \int \arcsin \frac{x}{a} dx = x \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}$$

$$493. \int x \arcsin \frac{x}{a} dx = \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \arcsin \frac{x}{a} + \frac{x}{4} \sqrt{a^2 - x^2}$$

$$494. \int x^2 \arcsin \frac{x}{a} dx = \frac{x^3}{3} \arcsin \frac{x}{a} + \frac{1}{9} (x^2 + 2a^2) \sqrt{a^2 - x^2}$$

$$495. \int x^n \arcsin \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \arcsin \frac{x}{a} - \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{a^2 - x^2}} dx \quad (n \neq -1)$$

$$496. \int \frac{\arcsin \frac{x}{a} dx}{x} = \frac{x}{a} + \frac{1}{2 \cdot 3 \cdot 3 a^3}$$

$$+ \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 5 a^5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 7 a^7} + \dots$$

($x^2 < a^2$)

$$497. \int \frac{\arcsin \frac{x}{a} dx}{x^2} = -\frac{1}{x} \arcsin \frac{x}{a} - \frac{1}{a} \log_e \frac{a + \sqrt{a^2 - x^2}}{x}$$

$$498. \int \left(\arcsin \frac{x}{a} \right)^2 dx = x \left(\arcsin \frac{x}{a} \right)^2 + 2 \left(\sqrt{a^2 - x^2} \arcsin \frac{x}{a} - x \right)$$

$$499. \int \arccos \frac{x}{a} dx = x \arccos \frac{x}{a} - \sqrt{a^2 - x^2}$$

$$500. \int x \arccos \frac{x}{a} dx = \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \arccos \frac{x}{a} - \frac{x}{4} \sqrt{a^2 - x^2}$$

$$501. \int x^2 \arccos \frac{x}{a} dx = \frac{x^3}{3} \arccos \frac{x}{a} - \frac{1}{9} (x^2 + 2a^2) \sqrt{a^2 - x^2}$$

$$502. \int x^n \arccos \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \arccos \frac{x}{a} + \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{a^2 - x^2}} dx$$

($n \neq -1$)

$$503. \int \frac{\arccos \frac{x}{a} dx}{x} = \frac{\pi}{2} \log_e x - \frac{x}{a}$$

$$- \frac{1}{2 \cdot 3 \cdot 3 a^3} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 5 a^5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 7 a^7} - \dots$$

($x^2 < a^2$)

$$504. \int \frac{\arccos \frac{x}{a} dx}{x^2} = -\frac{1}{x} \arccos \frac{x}{a} + \frac{1}{a} \log_e \frac{a + \sqrt{a^2 - x^2}}{x}$$

$$505. \int \left(\arccos \frac{x}{a} \right)^2 dx = x \left(\arccos \frac{x}{a} \right)^2 - 2 \left(\sqrt{a^2 - x^2} \arccos \frac{x}{a} + x \right)$$

$$506. \int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \log_e (a^2 + x^2)$$

$$507. \int x \arctan \frac{x}{a} dx = \frac{1}{2} (x^2 + a^2) \arctan \frac{x}{a} - \frac{ax}{2}$$

$$508. \int x^2 \arctan \frac{x}{a} dx = \frac{x^3}{3} \arctan \frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6} \log_e (a^2 + x^2)$$

$$509. \int x^n \arctan \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \arctan \frac{x}{a} - \frac{a}{n+1} \int \frac{x^{n+1} dx}{a^2 + x^2}$$

($n \neq -1$)

510. $\int \frac{\arctan \frac{x}{a} dx}{x} = \frac{x}{a} - \frac{x^3}{3^2 a^3} + \frac{x^5}{5^2 a^5} - \frac{x^7}{7^2 a^7} + \dots \quad (|x| < |a|)$
511. $\int \frac{\arctan \frac{x}{a} dx}{x^2} = -\frac{1}{x} \arctan \frac{x}{a} - \frac{1}{2a} \log_e \frac{a^2 + x^2}{x^2}$
512. $\int \frac{\arctan \frac{x}{a} dx}{x^n} = -\frac{1}{(n-1)x^{n-1}} \arctan \frac{x}{a} + \frac{a}{n-1} \int \frac{dx}{x^{n-1}(a^2 + x^2)} \quad (n \neq 1)$
513. $\int \operatorname{arccot} \frac{x}{a} dx = x \operatorname{arccot} \frac{x}{a} + \frac{a}{2} \log_e (a^2 + x^2)$
514. $\int x \operatorname{arccot} \frac{x}{a} dx = \frac{1}{2} (x^2 + a^2) \operatorname{arccot} \frac{x}{a} + \frac{ax}{2}$
515. $\int x^2 \operatorname{arccot} \frac{x}{a} dx = \frac{x^3}{3} \operatorname{arccot} \frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6} \log_e (a^2 + x^2)$
516. $\int x^n \operatorname{arccot} \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \operatorname{arccot} \frac{x}{a} + \frac{a}{n+1} \int \frac{x^{n+1} dx}{a^2 + x^2} \quad (n \neq -1)$
517. $\int \frac{\operatorname{arccot} \frac{x}{a} dx}{x} = \frac{\pi}{2} \log_e x - \frac{x}{a} + \frac{x^3}{3^2 a^3} - \frac{x^5}{5^2 a^5} + \frac{x^7}{7^2 a^7} - \dots$
518. $\int \frac{\operatorname{arccot} \frac{x}{a} dx}{x^2} = -\frac{1}{x} \operatorname{arccot} \frac{x}{a} + \frac{1}{2a} \log_e \frac{a^2 + x^2}{x^2}$
519. $\int \frac{\operatorname{arccot} \frac{x}{a} dx}{x^n} = -\frac{1}{(n-1)x^{n-1}} \operatorname{arccot} \frac{x}{a} - \frac{a}{n-1} \int \frac{dx}{x^{n-1}(a^2 + x^2)} \quad (n \neq 1)$

$$520. \int \sinh^{-1} \frac{x}{a} dx = x \sinh^{-1} \frac{x}{a} - \sqrt{x^2 + a^2}$$

$$521. \int x \sinh^{-1} \frac{x}{a} dx = \frac{1}{2} \left(x^2 + \frac{a^2}{2} \right) \sinh^{-1} \frac{x}{a} - \frac{x}{4} \sqrt{x^2 + a^2}$$

$$522. \int \cosh^{-1} \frac{x}{a} dx = x \cosh^{-1} \frac{x}{a} \mp \sqrt{x^2 - a^2}$$

(upper sign for $\cosh^{-1} \frac{x}{a} > 0$)

$$523. \int \tanh^{-1} \frac{x}{a} dx = x \tanh^{-1} \frac{x}{a} + \frac{a}{2} \log_e (a^2 - x^2)$$

$$524. \int x \tanh^{-1} \frac{x}{a} dx = \frac{x^2 - a^2}{2} \tanh^{-1} \frac{x}{a} + \frac{ax}{2}$$

$$525. \int \coth^{-1} \frac{x}{a} dx = x \coth^{-1} \frac{x}{a} + \frac{a}{2} \log_e (x^2 - a^2)$$