Name: KEY

1. Given the vectors **u** and **v**, find $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{v}$.

$$\mathbf{u}=\langle -8,6,2\rangle,\quad \mathbf{v}=\langle 6,-3,-4\rangle$$

$\mathbf{u}\times\mathbf{v}$

$$\begin{vmatrix} i & j & k \\ -8 & 6 & 2 \\ 6 & -3 & -4 \end{vmatrix} = (-24+6)\vec{i}+(12-32)\vec{j}+(24-36)\vec{k}$$
$$= -18\vec{i}-20\vec{j}-12\vec{k}$$

$\mathbf{v}\times\mathbf{v} = \langle 0,0,0\rangle$

2. Given the vectors **u** nd **v**, find the cross product and determine whether it is orthogonal to both **u** and **v**.

$$\mathbf{u}=\langle -1,8,2\rangle,\quad \mathbf{v}=\langle 4,10,5\rangle$$

$\mathbf{u}\times\mathbf{v}$ $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u}$ $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}$

$$\begin{vmatrix} i & j & k \\ -1 & 8 & 2 \\ 4 & 10 & 5 \end{vmatrix} = (40-20)i+(8+5)j+(-10-32)k$$
$$\langle -20,13,-42\rangle$$

$(u\times v)\cdot u =$
$\langle 20,13,-42\rangle\cdot\langle -1,8,2\rangle$
$=-20+8\cdot 13-84=0$

$(u\times v)\cdot v =$
$\langle +20,13,-42\rangle\cdot\langle 4,10,5\rangle$
$= -80+130-210=0$

3. Find the area of a parallelogram that has the given vectors as adjacent sides.

$$\mathbf{u}=\langle -2,5,2\rangle,\quad \mathbf{v}=\langle 6,2,3\rangle$$

area $= \|u\times v\|$

$$\begin{vmatrix} i & j & k \\ -2 & 5 & 2 \\ 6 & 2 & 3 \end{vmatrix} = \|\langle 15-4, 12+6, -4-30\rangle\|$$

$\|\langle 11,18,-34\rangle\|$
$=\sqrt{11^2+18^2+34^2}$
$=\sqrt{121+324+1156}=\sqrt{1601}$

4. Find the triple scalar product of the vectors

$$\mathbf{u}=\langle -6,7,5\rangle,\quad \mathbf{v}=\langle 5,6,-3\rangle,\quad \mathbf{w}=\langle -4,0,-7\rangle$$

$v\times w$

$$\begin{vmatrix} i & j & k \\ 5 & 6 & -3 \\ -4 & 0 & -7 \end{vmatrix} = \langle -42, 12+35, 24\rangle$$

$u\cdot v\times w = \langle -6,7,5\rangle\cdot\langle -42,47,24\rangle$

$252+329+120=701$

5. **Use the triple scalar product to find the volume of the parallelepiped having adjacent edges given by the vectors**

$\mathbf{u} = \langle 3,7,2\rangle$, $\mathbf{v} = \langle 0,9,4\rangle$, $\mathbf{w} = \langle 2,8,-2\rangle$

$u \times v = \begin{vmatrix} i & j & k \\ 3 & 7 & 2 \\ 0 & 9 & 4 \end{vmatrix} = \langle 28-18, -12, 27\rangle = \langle 10, -12, 27\rangle$

$(u \times v) \cdot w$

$|\langle 10,-12,27\rangle \cdot \langle 2,8,-2\rangle|$

$|20-96-54| = |-130| = 130$

6. **Find a set of parametric equations of the line through the point $(-6,9,4)$ parallel to the vector v=(6, 8,2).**

$\langle x, y, z\rangle = \langle -6, 9, 4\rangle + t\langle 6, 8, 2\rangle$

$x = -6 + 6t$
$y = 9 + 8t$
$z = 4 + 2t$

7. **Find a set of symmetric equations of the line through the point $(7,9,4)$ parallel to the vector v=(6, 6,8).**

$\langle x, y, z\rangle = \langle 7, 9, 4\rangle + t\langle 6, 6, 8\rangle$

$x = 7 + 6t$
$y = 9 + 6t$
$z = 4 + 8t$

$\Rightarrow t = \frac{x-7}{6}$, $t = \frac{y-9}{6}$, $t = \frac{z-4}{8}$ $\Rightarrow \frac{x-7}{6} = \frac{y-9}{6} = \frac{z-4}{8}$

8. **Find a set of parametric equations of the line through the points $(-7,6,4)$ and $(-17,2,-10)$.**

direction vector $= \langle -7,6,4\rangle - \langle -17,2,-10\rangle$
$= \langle 10, 4, 14\rangle$

$\langle x, y, z\rangle = \langle -7, 6, 4\rangle + t\langle 10, 4, 14\rangle$

$x = -7 + 10t$
$y = 6 + 4t$
$z = 4 + 14t$

9. **Find a set of symmetric equations of the line through the points $(8,5,4)$ and $(1,3,-2)$.**

direction vector $= \langle 8,5,4\rangle - \langle 1,3,-2\rangle$
$= \langle 7, 2, 6\rangle$

$\langle x, y, z\rangle = \langle 8,5,4\rangle + t\langle 7,2,6\rangle$

$x = 8 + 7t$
$y = 5 + 2t$
$z = 4 + 6t$

$\Rightarrow t = \frac{x-8}{7}$, $t = \frac{y-5}{2}$, $t = \frac{z-4}{6}$ $\Rightarrow \frac{x-8}{7} = \frac{y-5}{2} = \frac{z-4}{6}$

10. **Find the set of parametric equations of the line through the point $(-8,8,3)$ and is parallel to the line $x=2+8t$, $y=9+8t$, and $z=-2+6t$.**

direction vector $= \langle 8, 8, 6\rangle$
or $\langle 4, 4, 3\rangle$

$\langle x, y, z\rangle = \langle -8, 8, 3\rangle + t\langle 4, 4, 3\rangle$

$x = -8 + 4t$
$y = 8 + 4t$
$z = 3 + 3t$

11. Determine whether any of the lines given below are parallel or identical.

direction vector

$L_1:\ x=-7-4t,\ y=3-8t,\ z=-4-7t$ $\langle -4,-8,-7\rangle = (-1)\langle 4,8,7\rangle$

$L_2:\ x=1+8t,\ y=19+16t,\ z=10+14t$ $\langle 8,16,14\rangle = 2\langle 4,8,7\rangle$

$L_3:\ x=4t,\ y=2-8t,\ z=1-7t$ $\langle 4,-8,-7\rangle \neq c\langle 4,8,7\rangle$

$L_4:\ x=1-8t,\ y=19-16t,\ z=10-14t$ $\langle -8,-16,-14\rangle = -2\langle 4,8,7\rangle$

So L_1, L_2 and L_4 are parallel

when $t=0$ both L_2 and L_4 contain $(1,19,10)$ so they are the same line.

when $t=-2$, L_1 contain $(1,19,10)$ also. So L_1, L_2 and L_4 are the same line.

12. Determine whether any of the lines given below are parallel or identical.

direction vector

$L_1:\ \dfrac{x-4}{2}=\dfrac{y-4}{8}=\dfrac{z-7}{4}$ $\langle 2,8,4\rangle = 2\langle 1,4,2\rangle$

$L_2:\ \dfrac{x-1}{-6}=\dfrac{y-7}{-24}=\dfrac{z-10}{12}$ $\langle -6,-24,12\rangle = -6\langle 1,4,-2\rangle$

parallel

$L_3:\ \dfrac{x}{2}=\dfrac{y-2}{-8}=\dfrac{z-1}{-4}$ $\langle 2,-8,-4\rangle = 2\langle 1,-4,-2\rangle$

$L_4:\ \dfrac{x-1}{6}=\dfrac{y-7}{24}=\dfrac{z-10}{-12}$ $\langle 6,24,-12\rangle = 6\langle 1,4,-2\rangle$

Note that L_2 and L_4 both contain $(1,7,10)$ and are parallel

So they are the same line.

13. Determine whether the lines given below meet. and, if so, where.

$x=-8+7t,\ y=8+4t,\ z=-3+2t$

$x=2+3s,\ y=14+2s,\ z=2+3s$

~~t=1 s=-1~~

At the same point

$-8+7t=2+3s$ $\quad\Rightarrow\quad$ $7t-3s=10$

$8+4t=14+2s$ $\quad\Rightarrow\quad$ $4t-2s=6$

$-3+2t=2+3s$ $\quad\Rightarrow\quad$ $2t-3s=5$

solve eqn 2 for $s=2t-3$ then plug into eqn 1 & 3

$7t-3(2t-3)=10 \Rightarrow t+9=10 \Rightarrow t=1 \Rightarrow s=2\cdot 1-3=-1$

$2t-3(2t-3)=5 \Rightarrow -4t+9=5 \Rightarrow t=1$

So the lines meet at

$(-8+7\cdot 1,\ 8+4\cdot 1,\ -3+2\cdot 1) = (-1,12,-1) = (2+3(-1),\ 14+2(-1),\ 2+3(-1))$

14. Determine whether the lines given below are parallel or where they meet.

$$\frac{x-7}{4}=\frac{y-8}{2}=\frac{z-0}{8} \quad \langle 4,2,8\rangle$$
$$\frac{x-5}{2}=\frac{y-19}{-3}=\frac{z-2}{2} \quad \langle 2,-3,2\rangle$$

direction vector; not parallel

In parametric form:

L_1: $x=4t+7,\ y=2t+8,\ z=8t$

L_2: $x=2s+5,\ y=-3s+19,\ z=2s+2$

at any intersection

$$\begin{cases}4t+7=2s+5\\ 2t+8=-3s+19\\ 8t=2s+2\end{cases} \Rightarrow \begin{cases}4t=2s-2\\ 4t=-6s+22\\ 4t=s+1\end{cases}$$

$2s-2=s+1 \Rightarrow s=3 \Rightarrow t=1$

so intersection test

$L_1(t=1)$: $(4+7, 2+8, 8)=(11,10,8)$

$L_2(s=3)$: $(6+5, -9+19, 6+2)=(11,10,8)$

THEY MEET!

15. Find an equation of a plane passing through the point given and perpendicular to the given vector.

Point: (1,6,6) Vector $\mathbf{v}=\langle 3,6,3\rangle$

$$\underbrace{(\langle x,y,z\rangle-\langle 1,6,6\rangle)}_{\text{vector in plane}}\cdot\underbrace{\langle 3,6,3\rangle}_{\text{normal vector}}=0 \Rightarrow 3(x-1)+6(y-6)+3(z-6)=0$$

16. Find an equation of a plane passing through the following three points.

(–3,–1,–13), (5,3,3), (–2,0,–12)

normal vector: $=\overrightarrow{(-3,-1,-13)(5,3,3)}\times\overrightarrow{(-2,0,-12)(5,3,3)}$

$=\langle 8,4,16\rangle\times\langle 7,3,15\rangle=\begin{vmatrix} i & j & k\\ 8 & 4 & 16\\ 7 & 3 & 15\end{vmatrix}$

$\rightarrow \langle 60-48, 112-120, 24-28\rangle$

$=\langle 12,-8,-4\rangle$

or equivalently

$\vec{n}=\langle 3,-2,-1\rangle$

Equation $3(x-5)-2(y-3)-(z-3)=0$

17. Find an equation of a plane passing through the points

(–1,1,1), (1,2,6)

and perpendicular to the plane

$2x+y+5z+7=0$.

The normal vector for the new plane must be orthogonal to both

$\overrightarrow{(-1,1,1)(1,2,6)}=\langle 2,1,5\rangle$

and (orthogonal to) the normal vector of $2x+y+5z+7=0$, also $\langle 2,1,5\rangle$.

So Let $\vec{n}=\langle -1,2,0\rangle$

(or $\langle 5,0,-2\rangle$ or $\langle 0,5,-1\rangle$ or etc.)

Eqn:

$-(x+1)+2(y-1)+0(z-1)=0$

$\equiv\ 2y-x=3$

18. Determine whether the following planes are parallel, orthogonal, or neither. If they are neither parallel nor orthogonal, find the angle of intersection.

normal vectors

$-4x-0y+4z+4=0$ $\langle -4,0,4\rangle = \vec{n}_1$

$2x+y+2z-4=0$ $\langle 2,1,2\rangle = \vec{n}_2$

$\vec{n}_1 \cdot \vec{n}_2 = -8+0+8=0$

$\Rightarrow$ planes are orthogonal

19. Determine whether the following planes are parallel, orthogonal, or neither. If they are neither parallel nor orthogonal, find the angle of intersection.

normal vectors

$-2x-7y+4z+2=0$ $\langle -2,-7,4\rangle = \vec{n}_1$

$2x+8y-8z-4=0$ $\langle 2,8,-8\rangle = \vec{n}_2$

$\vec{n}_1 \cdot \vec{n}_2 = -92 \Rightarrow$ not orthogonal

$\vec{n}_1 \neq c\vec{n}_2$ for any $c \in \mathbb{R} \Rightarrow$ not parallel

$\cos\theta = \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\,\|\vec{n}_2\|} = \dfrac{92}{\sqrt{4+49+16}\sqrt{4+64+64}}$

$= \dfrac{92}{\sqrt{69}\sqrt{132}}$

$\Rightarrow \theta = \arccos(.964)$

$= .269$ radians

$\doteq 15.42^\circ$

20. Find the distance between the point (1, 2, 3) and the plane given below.

$\vec{n} = \langle 5,-8,7\rangle$

$5x-8y+7z=14$

point in the plane

$(0,0,2)$

vector to plane: $\overrightarrow{(1\,2\,3)(0,0,2)} = \langle 1,2,1\rangle = \vec{v}$

distance to plane

$\dfrac{|\vec{v}\cdot\vec{n}|}{\|\vec{n}\|} = \dfrac{|\langle 1,2,1\rangle \cdot \langle 5,-8,7\rangle|}{\sqrt{25+64+49}} = \dfrac{4}{\sqrt{138}} \approx .3405$

21. Find the distance between the planes given below.

normal vector point in Plane

P_1: $7x-4y+2z-5=0$ $\langle 7,-4,2\rangle = \vec{n}$ $(1,0,-1)$

P_2: $14x-8y+4z-16=0$ " $(0,-2,0)$

vector between planes: $\overrightarrow{(1,0,-1)\ (0,-2,0)} = \langle 1,2,-1\rangle = \vec{v}$

distance between Planes:

$\dfrac{|\vec{v}\cdot\vec{n}|}{\|\vec{n}\|} = \dfrac{|\langle 1,2,-1\rangle \cdot \langle 7,-4,2\rangle|}{\sqrt{49+16+4}} = \dfrac{|7-8-2|}{\sqrt{69}} = \dfrac{3}{\sqrt{69}} \approx .3612$

22. Identify the following quadratic surface.

$$\frac{x^2}{3}+\frac{y^2}{14}+\frac{z^2}{4}=1$$

intersecting coordinate planes:

$x=0$: $\frac{y^2}{14}+\frac{z^2}{4}=1$: ellipse

$y=0$: $\frac{x^2}{2}+\frac{z^2}{4}=1$: ellipse

$z=0$: $\frac{x^2}{3}+\frac{y^2}{14}=1$

$\Rightarrow$ ellipsoid

23. Identify the following quadratic surface.

$$\frac{x^2}{4}+\frac{y^2}{8}-\frac{z^2}{16}=1$$

intersecting coordinate planes

$x=0$: $\frac{y^2}{8}-\frac{z^2}{16}=1$: hyperbola

$y=0$: $\frac{x^2}{4}-\frac{z^2}{16}=1$: hyperbola

$z=0$: $\frac{x^2}{4}+\frac{y^2}{8}=1$: ellipse

$\Rightarrow$ hyperboloid of one sheet

24. Identify the following quadratic surface.

$$\frac{x^2}{4}-\frac{y^2}{6}-\frac{z^2}{12}=1$$

intersecting planes

$x=0$: $-\frac{y^2}{6}-\frac{z^2}{12}=1$ none

$y=0$: $\frac{x^2}{4}-\frac{z^2}{12}=1$ hyperbola

$z=0$ $\frac{x^2}{4}-\frac{z^2}{12}=1$ hyperbola

$\Rightarrow$ hyperboloid of two sheets

25. Identify the following quadratic surface.

$$\frac{x^2}{2}+\frac{y^2}{16}-\frac{z^2}{10}=0$$

intersecting planes

$x=0$: $\frac{y^2}{16}-\frac{z^2}{10}=0$ crossed lines

$y=0$: $\frac{x^2}{2}-\frac{z^2}{10}=0$ crossed lines

$z=0$: $\frac{x^2}{2}+\frac{y^2}{16}=0$ point

$\Rightarrow$ elliptic cone

26. Identify the following quadratic surface.

$$z=\frac{x^2}{4}+\frac{y^2}{4}$$

intersecting planes

$x=0$: $z=\frac{y^2}{4}$ parabola

$y=0$: $z=\frac{x^2}{4}$ parabola

$z=1$: $1=\frac{x^2}{4}+\frac{y^2}{4}$ circle

$\Rightarrow$ elliptic parabaloid

27. Identify the following quadratic surface.

$$z = \frac{x^2}{8} - \frac{y^2}{4}$$

intersectin planes

$x=0$: $z = -\frac{y^2}{4}$ parabola

$y=0$: $z = \frac{x^2}{8}$ parabola

$z=1$: $1 = \frac{x^2}{8} - \frac{y^2}{4}$ hyperbola

$\Rightarrow$ hyperbolic paraboloid

28. OMIT

29. OMIT

30. OMIT

31. OMIT

32. Convert the following point from cylindrical coordinates to rectangular coordinates.

$\left(8, \frac{\pi}{6}, 6\right)$

$x = r\cos\theta = 8\cos\left(\frac{\pi}{6}\right) = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}$

$y = r\sin\theta = 8\sin\left(\frac{\pi}{6}\right) = 8 \cdot \frac{1}{2} = 4$

$z = z = 6$

$(4\sqrt{3}, 4, 6)$

33. Convert the following point from rectangular coordinates to cylindrical coordinates. Give any angles in radians.

(4,1,4)

$r = \sqrt{x^2+y^2} = \sqrt{16+1} = \sqrt{17}$

$\tan\theta = \frac{1}{4} \Rightarrow \theta = .245$ radians

$z = 4$

$(\sqrt{17}, .245, 4)$

34. Find an equation in cylindrical coordinates for the equation given in rectangular coordinates.

$z = 49x^2 + 49y^2 - 4$

$z = 49(x^2+y^2) - 4$

$z = 49r^2 - 4$

35. Find an equation in cylindrical coordinates for the equation given in rectangular coordinates.

$9x^2 + 9y^2 = 2x$

$9(x^2+y^2) = 2x$

$9r^2 = 2r\cos\theta$

$r = \frac{2}{9}\cos\theta$

36. Find an equation in cylindrical coordinates for the equation given in rectangular coordinates.

$25x^2 + 25y^2 - 4z^2 = g$

$25(x^2+y^2) - 4z^2 = g$

$25 \; r^2 - 4z^2 = g$

$z^2 = \dfrac{25r^2 - g}{4}$

37. Find an equation in rectangular coordinates for the equation given in cylindrical coordinates.

$r = 5\sin\theta$

$r^2 = 5r\sin\theta$

$x^2+y^2 = 5y$

38. Find an equation in rectangular coordinates for the equation given in cylindrical coordinates

$r = 4z$

$r^2 = 16z^2$

$x^2+y^2 = 16z^2$

39. Find an equation in rectangular coordinates for the equation given in cylindrical coordinates.

$r^2 + z^2 = 25$

$x^2+y^2+z^2 = 25$

40. Convert the point from spherical coordinates to rectangular coordinates.

$\left(4, \frac{\pi}{6}, \frac{\pi}{16}\right)$

(ρ, θ, ϕ)

$z = \rho\cos\phi = 4\cos\frac{\pi}{16} = 3.923$

$r = \rho\sin\phi = 4\sin\frac{\pi}{16} = .7804$

$x = r\cos\theta = 4\sin\frac{\pi}{16}\cos\frac{\pi}{6} = 2\sqrt{3}\sin\frac{\pi}{16} = .6758$

$y = r\sin\theta = 4\sin\frac{\pi}{16}\sin\frac{\pi}{6} = 2\sin\frac{\pi}{16} = .3902$

$(.6758, .3902, 3.923)$

41. Find an equation in spherical coordinates for the equation given in rectangular coordinates.

$y=2$

$r\sin\theta=2$

$\rho\sin\phi\sin\theta=2$

solve for ρ

$\rho=2\csc\phi\csc\theta$

42. Find an equation in spherical coordinates for the equation given in rectangular coordinates.

$x^2+y^2-6z^2=3$

$r^2-6z^2=3$

$(\rho\sin\phi)^2-6(\rho\cos\phi)^2=3$

solve for ρ^2

$\rho^2[\sin^2\phi-6\cos^2\phi]=3$

$\rho^2=\dfrac{3}{\sin^2\phi-6\cos^2\phi}$

43. Find an equation in rectangular coordinates for the equation given in spherical coordinates.

$\theta=\dfrac{\pi}{8}$

$\tan\left(\frac{\pi}{8}\right)=\frac{y}{x} \Rightarrow y=x\tan\left(\frac{\pi}{8}\right)$ or $y=.414x$

44. Find an equation in rectangular coordinates for the equation given in spherical coordinates.

$\rho=3\csc\varphi\ \csc\theta$

$\rho=\dfrac{3}{\sin\phi\sin\theta}$

$\rho\sin\phi\sin\theta=3$

$r\sin\theta=3$

$\boxed{y=3}$

45. Convert the following point from cylindrical coordinates to spherical coordinates.

$\left(6,\frac{\pi}{3},8\right)$

(r,θ,z)

$\theta=\frac{\pi}{3}$

$\rho^2=r^2+z^2$

$\rho^2=36+64$

$\rho=10$

$\tan\phi=\frac{r}{z}=\frac{6}{8}=\frac{3}{4}$

$\phi=\arctan\left(\frac{3}{4}\right)=.6435$ radians

$(\rho,\theta,\phi)=\left(10,\frac{\pi}{3},.6435\right)$

46. Convert the following point from spherical coordinates to cylindrical coordinates.

$\left(7, \frac{\pi}{16}, \frac{\pi}{4}\right)$

(ρ, θ, ϕ)

$z = \rho \cos\phi = 7\cos\left(\frac{\pi}{4}\right) = \frac{7}{\sqrt{2}}$

$r = \rho \sin\phi = 7\sin\left(\frac{\pi}{4}\right) = \frac{7}{\sqrt{2}}$

$\theta = \theta = \frac{\pi}{16}$

$\left(\frac{7}{\sqrt{2}}, \frac{\pi}{16}, \frac{7}{\sqrt{2}}\right) = (r, \theta, z)$