Exercise Set 3.1*

1. A menagerie consists of seven brown dogs, two black dogs, six gray cats, ten black cats, five blue birds, six yellow birds, and one black bird. Determine which of the following statements are true and which are false.
   a. There is an animal in the menagerie that is red.
   b. Every animal in the menagerie is a bird or a mammal.
   c. Every animal in the menagerie is brown or gray or black.
   d. There is an animal in the menagerie that is neither a cat nor a dog.
   e. No animal in the menagerie is blue.
   f. There are in the menagerie a dog, a cat, and a bird that all have the same color.

2. Indicate which of the following statements are true and which are false. Justify your answers as best as you can.
   a. Every integer is a real number.
   b. 0 is a positive real number.
   c. For all real numbers r, $-r$ is a negative real number.
   d. Every real number is an integer.

3. Let $P(x)$ be the predicate "$x > 1/x$."
   a. Write $P(2)$, $P(\frac{1}{2})$, $P(-1)$, $P(-\frac{1}{2})$, and $P(-8)$, and indicate which of these statements are true and which are false.
   b. Find the truth set of $P(x)$ if the domain of x is **R**, the set of all real numbers.
   c. If the domain is the set $\mathbf{R}^+$ of all positive real numbers, what is the truth set of $P(x)$?

4. Let $Q(n)$ be the predicate "$n^2 \leq 30$."
   a. Write $Q(2)$, $Q(-2)$, $Q(7)$, and $Q(-7)$, and indicate which of these statements are true and which are false.
   b. Find the truth set of $Q(n)$ if the domain of n is **Z**, the set of all integers.
   c. If the domain is the set $\mathbf{Z}^+$ of all positive integers, what is the truth set of $Q(n)$?

5. Let $Q(x, y)$ be the predicate "If $x < y$ then $x^2 < y^2$" with domain for both x and y being the set **R** of real numbers.
   a. Explain why $Q(x, y)$ is false if $x = -2$ and $y = 1$.
   b. Give values different from those in part (a) for which $Q(x, y)$ is false.
   c. Explain why $Q(x, y)$ is true if $x = 3$ and $y = 8$.
   d. Give values different from those in part (c) for which $Q(x, y)$ is true.

6. Let $R(m, n)$ be the predicate "If m is a factor of n^2 then m is a factor of n," with domain for both m and n being the set **Z** of integers.
   a. Explain why $R(m, n)$ is false if $m = 25$ and $n = 10$.
   b. Give values different from those in part (a) for which $R(m, n)$ is false.
   c. Explain why $R(m, n)$ is true if $m = 5$ and $n = 10$.
   d. Give values different from those in part (c) for which $R(m, n)$ is true.

7. Find the truth set of each predicate.
   a. predicate: $6/d$ is an integer, domain: **Z**
   b. predicate: $6/d$ is an integer, domain: $\mathbf{Z}^+$
   c. predicate: $1 \leq x^2 \leq 4$, domain: **R**
   d. predicate: $1 \leq x^2 \leq 4$, domain: **Z**

8. Let $B(x)$ be "$-10 < x < 10$." Find the truth set of $B(x)$ for each of the following domains.
   a. **Z** b. $\mathbf{Z}^+$ c. The set of all even integers

Find counterexamples to show that the statements in 9–12 are false.

9. $\forall x \in \mathbf{R}, x > 1/x$.

10. $\forall a \in \mathbf{Z}$, $(a-1)/a$ is not an integer.

11. $\forall$ positive integers m and n, $m \cdot n \geq m + n$.

12. $\forall$ real numbers x and y, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

13. Consider the following statement:

$\forall$ basketball players x, x is tall.

Which of the following are equivalent ways of expressing this statement?
   a. Every basketball player is tall.
   b. Among all the basketball players, some are tall.
   c. Some of all the tall people are basketball players.
   d. Anyone who is tall is a basketball player.
   e. All people who are basketball players are tall.
   f. Anyone who is a basketball player is a tall person.

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ✱ signals that an exercise is more challenging than usual.

14. Consider the following statement:

$$\exists x \in \mathbf{R} \text{ such that } x^2 = 2.$$

Which of the following are equivalent ways of expressing this statement?

a. The square of each real number is 2.
b. Some real numbers have square 2.
c. The number x has square 2, for some real number x.
d. If x is a real number, then $x^2 = 2$.
e. Some real number has square 2.
f. There is at least one real number whose square is 2.

H 15. Rewrite the following statements informally in at least two different ways without using variables or quantifiers.

a. $\forall$ rectangles x, x is a quadrilateral.
b. $\exists$ a set A such that A has 16 subsets.

16. Rewrite each of the following statements in the form "$\forall$ ______ x, ______."

a. All dinosaurs are extinct.
b. Every real number is positive, negative, or zero.
c. No irrational numbers are integers.
d. No logicians are lazy.
e. The number 2,147,581,953 is not equal to the square of any integer.
f. The number -1 is not equal to the square of any real number.

17. Rewrite each of the following in the form "$\exists$ ______ x such that ______."

a. Some exercises have answers.
b. Some real numbers are rational.

18. Let D be the set of all students at your school, and let $M(s)$ be "s is a math major," let $C(s)$ be "s is a computer science student," and let $E(s)$ be "s is an engineering student." Express each of the following statements using quantifiers, variables, and the predicates $M(s)$, $C(s)$, and $E(s)$.

a. There is an engineering student who is a math major.
b. Every computer science student is an engineering student.
c. No computer science students are engineering students.
d. Some computer science students are also math majors.
e. Some computer science students are engineering students and some are not.

19. Consider the following statement:

$$\forall \text{ integers } n, \text{ if } n^2 \text{ is even then } n \text{ is even.}$$

Which of the following are equivalent ways of expressing this statement?

a. All integers have even squares and are even.
b. Given any integer whose square is even, that integer is itself even.
c. For all integers, there are some whose square is even.
d. Any integer with an even square is even.
e. If the square of an integer is even, then that integer is even.
f. All even integers have even squares.

H 20. Rewrite the following statement informally in at least two different ways without using variables or the symbol $\forall$ or the words "for all."

$\forall$ real numbers x, if x is positive, then
the square root of x is positive.

21. Rewrite the following statements so that the quantifier trails the rest of the sentence.

a. For any graph G, the total degree of G is even.
b. For any isosceles triangle T, the base angles of T are equal.
c. There exists a prime number p such that p is even.
d. There exists a continuous function f such that f is not differentiable.

22. Rewrite each of the following statements in the form "$\forall$ ______ x, if ______ then ______."

a. All Java programs have at least 5 lines.
b. Any valid argument with true premises has a true conclusion.

23. Rewrite each of the following statements in the two forms "$\forall x$, if ______ then ______" and "$\forall$ ______ x, ______" (without an if-then).

a. All equilateral triangles are isosceles.
b. Every computer science student needs to take data structures.

24. Rewrite the following statements in the two forms "$\exists$ ______ x such that ______" and "$\exists x$ such that ______ and ______."

a. Some hatters are mad. b. Some questions are easy.

25. The statement "The square of any rational number is rational" can be rewritten formally as "For all rational numbers x, x^2 is rational" or as "For all x, if x is rational then x^2 is rational." Rewrite each of the following statements in the two forms "$\forall$ ______ x, ______" and "$\forall x$, if ______, then ______" or in the two forms "$\forall$ ______ x and y, ______" and "$\forall x$ and y, if ______, then ______."

a. The reciprocal of any nonzero fraction is a fraction.
b. The derivative of any polynomial function is a polynomial function.
c. The sum of the angles of any triangle is 180°.
d. The negative of any irrational number is irrational.
e. The sum of any two even integers is even.
f. The product of any two fractions is a fraction.

26. Consider the statement "All integers are rational numbers but some rational numbers are not integers."

a. Write this statement in the form "$\forall x$, if ______ then ______, but $\exists$ ______ x such that ______."
b. Let Ratl(x) be "x is a rational number" and Int(x) be "x is an integer." Write the given statement formally using only the symbols Ratl(x), Int(x), $\forall$, $\exists$, $\wedge$, $\vee$, $\sim$, and $\rightarrow$.

27. Refer to the picture of Tarski's world given in Example 3.1.13. Let Above(x, y) mean that x is above y (but possibly in a different column). Determine the truth or falsity

of each of the following statements. Give reasons for your answers.

a. $\forall u$, Circle$(u) \to$ Gray(u).
b. $\forall u$, Gray$(u) \to$ Circle(u).
c. $\exists y$ such that Square$(y) \wedge$ Above(y, d).
d. $\exists z$ such that Triangle$(z) \wedge$ Above(f, z).

In 28–30, rewrite each statement without using quantifiers or variables. Indicate which are true and which are false, and justify your answers as best as you can.

28. Let the domain of x be the set D of objects discussed in mathematics courses, and let Real(x) be "x is a real number," Pos(x) be "x is a positive real number," Neg(x) be "x is a negative real number," and Int(x) be "x is an integer."
a. Pos(0)
b. $\forall x$, Real$(x) \wedge$ Neg$(x) \to$ Pos$(-x)$.
c. $\forall x$, Int$(x) \to$ Real(x).
d. $\exists x$ such that Real$(x) \wedge \sim$Int(x).

29. Let the domain of x be the set of geometric figures in the plane, and let Square(x) be "x is a square" and Rect(x) be "x is a rectangle."
a. $\exists x$ such that Rect$(x) \wedge$ Square(x).
b. $\exists x$ such that Rect$(x) \wedge \sim$Square(x).
c. $\forall x$, Square$(x) \to$ Rect(x).

30. Let the domain of x be the set $\mathbf{Z}$ of integers, and let Odd(x) be "x is odd," Prime(x) be "x is prime," and Square(x) be "x is a perfect square." (An integer n is said to be a **perfect square** if, and only if, it equals the square of some integer. For example, 25 is a perfect square because $25 = 5^2$.)
a. $\exists x$ such that Prime$(x) \wedge \sim$Odd(x).
b. $\forall x$, Prime$(x) \to \sim$Square(x).
c. $\exists x$ such that Odd$(x) \wedge$ Square(x).

H 31. In any mathematics or computer science text other than this book, find an example of a statement that is universal but is implicitly quantified. Copy the statement as it appears and rewrite it making the quantification explicit. Give a complete citation for your example, including title, author, publisher, year, and page number.

32. Let $\mathbf{R}$ be the domain of the predicate variable x. Which of the following are true and which are false? Give counter examples for the statements that are false.
a. $x > 2 \Rightarrow x > 1$
b. $x > 2 \Rightarrow x^2 > 4$
c. $x^2 > 4 \Rightarrow x > 2$
d. $x^2 > 4 \Leftrightarrow |x| > 2$

33. Let $\mathbf{R}$ be the domain of the predicate variables a, b, c, and d. Which of the following are true and which are false? Give counterexamples for the statements that are false.
a. $a > 0$ and $b > 0 \Rightarrow ab > 0$
b. $a < 0$ and $b < 0 \Rightarrow ab < 0$
c. $ab = 0 \Rightarrow a = 0$ or $b = 0$
d. $a < b$ and $c < d \Rightarrow ac < bd$