Exercise Set 3.3

1. Let C be the set of cities in the world, let N be the set of nations in the world, and let $P(c, n)$ be "c is the capital city of n." Determine the truth values of the following statements.
   a. P(Tokyo, Japan) b. P(Athens, Egypt)
   c. P(Paris, France) d. P(Miami, Brazil)

2. Let $G(x, y)$ be "$x^2 > y$." Indicate which of the following statements are true and which are false.
   a. $G(2, 3)$ b. $G(1, 1)$
   c. $G\left(\frac{1}{2}, \frac{1}{2}\right)$ d. $G(-2, 2)$

3. The following statement is true: "$\forall$ nonzero numbers x, $\exists$ a real number y such that $xy = 1$." For each x given below, find a y to make the predicate "$xy = 1$" true.
   a. $x = 2$ b. $x = -1$ c. $x = 3/4$

4. The following statement is true: "$\forall$ real numbers x, $\exists$ an integer n such that $n > x$."* For each x given below, find an n to make the predicate "$n > x$" true.
   a. $x = 15.83$ b. $x = 10^8$ c. $x = 10^{10^{10}}$

The statements in exercises 5–8 refer to the Tarski world given in Example 3.3.1. Explain why each is true.

5. For all circles x there is a square y such that x and y have the same color.

6. For all squares x there is a circle y such that x and y have different colors and y is above x.

7. There is a triangle x such that for all squares y, x is above y.

8. There is a triangle x such that for all circles y, y is above x.

9. Let $D = E = \{-2, -1, 0, 1, 2\}$. Explain why the following statements are true.
   a. $\forall x$ in D, $\exists y$ in E such that $x + y = 0$.
   b. $\exists x$ in D such that $\forall y$ in E, $x + y = y$.

10. This exercise refers to Example 3.3.3. Determine whether each of the following statements is true or false.
    a. $\forall$ students S, $\exists$ a dessert D such that S chose D.
    b. $\forall$ students S, $\exists$ a salad T such that S chose T.
    c. $\exists$ a dessert D such that $\forall$ students S, S chose D.
    d. $\exists$ a beverage B such that $\forall$ students D, D chose B.
    e. $\exists$ an item I such that $\forall$ students S, S did not choose I.
    f. $\exists$ a station Z such that $\forall$ students S, $\exists$ an item I such that S chose I from Z.

11. Let S be the set of students at your school, let M be the set of movies that have ever been released, and let $V(s, m)$ be "student s has seen movie m." Rewrite each of the following statements without using the symbol $\forall$, the symbol $\exists$, or variables.
    a. $\exists s \in S$ such that $V(s, \text{Casablanca})$.
    b. $\forall s \in S$, $V(s, \text{Star Wars})$.
    c. $\forall s \in S$, $\exists m \in M$ such that $V(s, m)$.
    d. $\exists m \in M$ such that $\forall s \in S$, $V(s, m)$.
    e. $\exists s \in S$, $\exists t \in S$, and $\exists m \in M$ such that $s \neq t$ and $V(s, m) \wedge V(t, m)$.
    f. $\exists s \in S$ and $\exists t \in S$ such that $s \neq t$ and $\forall m \in M$, $V(s, m) \rightarrow V(t, m)$.

12. Let $D = E = \{-2, -1, 0, 1, 2\}$. Write negations for each of the following statements and determine which is true, the given statement or its negation.
    a. $\forall x$ in D, $\exists y$ in E such that $x + y = 1$.
    b. $\exists x$ in D such that $\forall y$ in E, $x + y = -y$.
    c. $\forall x$ in D, $\exists y$ in E such that $xy \geq y$.
    d. $\exists x$ in D such that $\forall y$ in E, $x \leq y$.

In each of 13–19, (a) rewrite the statement in English without using the symbol $\forall$ or $\exists$ or variables and expressing your answer as simply as possible, and (b) write a negation for the statement.

13. $\forall$ colors C, $\exists$ an animal A such that A is colored C.

14. $\exists$ a book b such that $\forall$ people p, p has read b.

15. $\forall$ odd integers n, $\exists$ an integer k such that $n = 2k + 1$.

16. $\exists$ a real number u such that $\forall$ real numbers v, $uv = v$.

17. $\forall r \in \mathbf{Q}$, $\exists$ integers a and b such that $r = a/b$.

18. $\forall x \in \mathbf{R}$, $\exists$ a real number y such that $x + y = 0$.

19. $\exists x \in \mathbf{R}$ such that for all real numbers y, $x + y = 0$.

20. Recall that reversing the order of the quantifiers in a statement with two different quantifiers may change the truth value of the statement—but it does not necessarily do so. All the statements in the pairs on the next page refer to the Tarski world of Figure 3.3.1. In each pair, the order of the quantifiers is reversed but everything else is the same. For each pair, determine whether the statements have the same or opposite truth values. Justify your answers.

*This is called the Archimedean principle because it was first formulated (in geometric terms) by the great Greek mathematician Archimedes of Syracuse, who lived from about 287 to 212 B.C.E.

a. (1) For all squares y there is a triangle x such that x and y have different color.
(2) There is a triangle x such that for all squares y, x and y have different colors.

b. (1) For all circles y there is a square x such that x and y have the same color.
(2) There is a square x such that for all circles y, x and y have the same color.

21. For each of the following equations, determine which of the following statements are true:
(1) For all real numbers x, there exists a real number y such that the equation is true.
(2) There exists a real number x, such that for all real numbers y, the equation is true.

Note that it is possible for both statements to be true or for both to be false.

a. $2x + y = 7$
b. $y + x = x + y$
c. $x^2 - 2xy + y^2 = 0$
d. $(x - 5)(y - 1) = 0$
e. $x^2 + y^2 = -1$

In 22 and 23, rewrite each statement without using variables or the symbol $\forall$ or $\exists$. Indicate whether the statement is true or false.

22. a. $\forall$ real numbers x, $\exists$ a real number y such that $x + y = 0$.
b. $\exists$ a real number y such that $\forall$ real numbers x, $x + y = 0$.

23. a. $\forall$ nonzero real numbers r, $\exists$ a real number s such that $rs = 1$.
b. $\exists$ a real number r such that $\forall$ nonzero real numbers s, $rs = 1$.

24. Use the laws for negating universal and existential statements to derive the following rules:

a. $\sim(\forall x \in D(\forall y \in E(P(x, y))))$
$$\equiv \exists x \in D(\exists y \in E(\sim P(x, y)))$$

b. $\sim(\exists x \in D(\exists y \in E(P(x, y))))$
$$\equiv \forall x \in D(\forall y \in E(\sim P(x, y)))$$

Each statement in 25–28 refers to the Tarski world of Figure 3.3.1. For each, (a) determine whether the statement is true or false and justify your answer, (b) write a negation for the statement (referring, if you wish, to the result in exercise 24).

25. $\forall$ circles x and $\forall$ squares y, x is above y.

26. $\forall$ circles x and $\forall$ triangles y, x is above y.

27. $\exists$ a circle x and $\exists$ a square y such that x is above y and x and y have different colors.

28. $\exists$ a triangle x and $\exists$ a square y such that x is above y and x and y have the same color.

For each of the statements in 29 and 30, (a) write a new statement by interchanging the symbols $\forall$ and $\exists$, and (b) state which is true: the given statement, the version with interchanged quantifiers, neither, or both.

29. $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}$ such that $x < y$.

30. $\exists x \in \mathbf{R}$ such that $\forall y \in \mathbf{R}^-$ (the set of negative real numbers), $x > y$.

31. Consider the statement "Everybody is older than somebody." Rewrite this statement in the form "$\forall$ people x, $\exists$ ______."

32. Consider the statement "Somebody is older than everybody." Rewrite this statement in the form "$\exists$ a person x such that $\forall$ ______."

In 33–39, (a) rewrite the statement formally using quantifiers and variables, and (b) write a negation for the statement.

33. Everybody loves somebody.

34. Somebody loves everybody.

35. Everybody trusts somebody.

36. Somebody trusts everybody.

37. Any even integer equals twice some integer.

38. Every action has an equal and opposite reaction.

39. There is a program that gives the correct answer to every question that is posed to it.

40. In informal speech most sentences of the form "There is ______ every ______" are intended to be understood as meaning "$\forall$ ______ $\exists$ ______," even though the existential quantifier *there is* comes before the universal quantifier *every*. Note that this interpretation applies to the following well-known sentences. Rewrite them using quantifiers and variables.

a. There is a sucker born every minute.
b. There is a time for every purpose under heaven.

41. Indicate which of the following statements are true and which are false. Justify your answers as best you can.

a. $\forall x \in \mathbf{Z}^+, \exists y \in \mathbf{Z}^+$ such that $x = y + 1$.
b. $\forall x \in \mathbf{Z}, \exists y \in \mathbf{Z}$ such that $x = y + 1$.
c. $\exists x \in \mathbf{R}$ such that $\forall y \in \mathbf{R}$, $x = y + 1$.
d. $\forall x \in \mathbf{R}^+, \exists y \in \mathbf{R}^+$ such that $xy = 1$.
e. $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}$ such that $xy = 1$.
f. $\forall x \in \mathbf{Z}^+$ and $\forall y \in \mathbf{Z}^+, \exists z \in \mathbf{Z}^+$ such that $z = x - y$.
g. $\forall x \in \mathbf{Z}$ and $\forall y \in \mathbf{Z}, \exists z \in \mathbf{Z}$ such that $z = x - y$.
h. $\exists u \in \mathbf{R}^+$ such that $\forall v \in \mathbf{R}^+$, $uv < v$.

42. Write the negation of the definition of limit of a sequence given in Example 3.3.7.

43. The following is the definition for $\lim_{x \to a} f(x) = L$:

For all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for all real numbers x, if $a - \delta < x < a + \delta$ and $x \neq a$ then $L - \varepsilon < f(x) < L + \varepsilon$.

Write what it means for $\lim_{x \to a} f(x) \neq L$. In other words, write the negation of the definition.

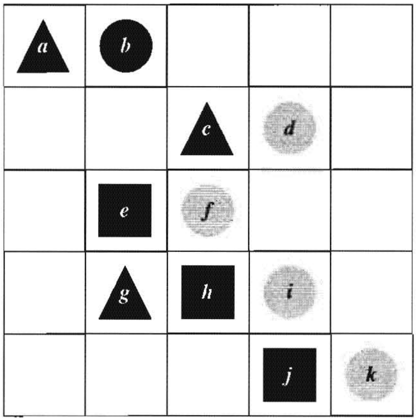
44. The notation $\exists!$ stands for the words "there exists a unique." Thus, for instance, "$\exists!$ x such that x is prime and x is even"

means that there is one and only one even prime number. Which of the following statements are true and which are false? Explain.

a. $\exists!$ real number x such that $\forall$ real numbers y, $xy = y$.
b. $\exists!$ integer x such that $1/x$ is an integer.
c. $\forall$ real numbers x, $\exists!$ real number y such that $x + y = 0$.

✶45. Suppose that $P(x)$ is a predicate and D is the domain of x. Rewrite the statement "$\exists!\ x \in D$ such that $P(x)$" without using the symbol $\exists!$. (See exercise 44 for the meaning of $\exists!$.)

In 46–54, refer to the Tarski world given in Figure 3.1.1, which is printed again here for reference. The domains of all variables consist of all the objects in the Tarski world. For each statement, (a) indicate whether the statement is true or false and justify your answer, (b) write the given statement using the formal logical notation illustrated in Example 3.3.10, and (c) write the negation of the given statement using the formal logical notation of Example 3.3.10.



46. There is a triangle x such that for all squares y, x is above y.

47. There is a triangle x such that for all circles y, x is above y.

48. For all circles x, there is a square y such that y is to the right of x.

49. For every object x, there is an object y such that $x \neq y$ and x and y have different colors.

50. For every object x, there is an object y such that if $x \neq y$ then x and y have different colors.

51. There is an object y such that for all objects x, if $x \neq y$ then x and y have different colors.

52. For all circles x and for all triangles y, x is to the right of y.

53. There is a circle x and there is a square y such that x and y have the same color.

54. There is a circle x and there is a triangle y such that x and y have the same color.

Let $P(x)$ and $Q(x)$ be predicates and suppose D is the domain of x. In 55–58, for the statement forms in each pair, determine whether (a) they have the same truth value for every choice of $P(x)$, $Q(x)$, and D, or (b) there is a choice of $P(x)$, $Q(x)$, and D for which they have opposite truth values.

55. $\forall x \in D, (P(x) \wedge Q(x))$, and
$(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$

56. $\exists x \in D, (P(x) \wedge Q(x))$, and
$(\exists x \in D, P(x)) \wedge (\exists x \in D, Q(x))$

57. $\forall x \in D, (P(x) \vee Q(x))$, and
$(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$

58. $\exists x \in D, (P(x) \vee Q(x))$, and
$(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

In 59–61, find the answers Prolog would give if the following questions were added to the program given in Example 3.3.11.

59. a. ?isabove(b_1, w_1)
b. ?color$(X$, white$)$
c. ?isabove(X, b_3)

60. a. ?isabove(w_1, g)
b. ?color$(w_2$, blue$)$
c. ?isabove(X, b_1)

61. a. ?isabove(w_2, b_3)
b. ?color$(X$, gray$)$
c. ?isabove(g, X)