Exercise Set 4.2

The numbers in 1–7 are all rational. Write each number as a ratio of two integers.

1. $-\frac{35}{6}$ 2. 4.6037 3. $\frac{4}{5}+\frac{2}{9}$

4. 0.37373737...

5. 0.56565656...

6. 320.5492492492...

7. 52.4672167216721...

8. The zero product property, says that if a product of two real numbers is 0, then one of the numbers must be 0.
   a. Write this property formally using quantifiers and variables.
   b. Write the contrapositive of your answer to part (a).
   c. Write an informal version (without quantifier symbols or variables) for your answer to part (b).

9. Assume that a and b are both integers and that $a \neq 0$ and $b \neq 0$. Explain why $(b-a)/(ab^2)$ must be a rational number.

10. Assume that m and n are both integers and that $n \neq 0$. Explain why $(5m+12n)/(4n)$ must be a rational number.

11. Prove that every integer is a rational number.

12. Fill in the blanks in the following proof that the square of any rational number is rational:

    Proof: Suppose that r is $\underline{\text{(a)}}$. By definition of rational, $r = a/b$ for some $\underline{\text{(b)}}$ with $b \neq 0$. By substitution,

    $$r^2 = \underline{\text{(c)}} = a^2/b^2.$$

    Since a and b are both integers, so are the products a^2 and $\underline{\text{(d)}}$. Also $b^2 \neq 0$ by the $\underline{\text{(e)}}$. Hence r^2 is a ratio of two integers with a nonzero denominator, and so $\underline{\text{(f)}}$ by definition of rational.

13. Consider the statement: The negative of any rational number is rational.
    a. Write the statement formally using a quantifier and a variable.
    b. Determine whether the statement is true or false and justify your answer.

14. Consider the statement: The square of any rational number is a rational number.
    a. Write the statement formally using a quantifier and a variable.
    b. Determine whether the statement is true or false and justify your answer.

Determine which of the statements in 15–20 are true and which are false. Prove each true statement directly from the definitions, and give a counterexample for each false statement.

In case the statement is false, determine whether a small change would make it true. If so, make the change and prove the new statement. Follow the directions for writing proofs on page 154.

15. The product of any two rational numbers is a rational number.

H **16.** The quotient of any two rational numbers is a rational number.

H **17.** The difference of any two rational numbers is a rational number.

H **18.** If r and s are any two rational numbers, then $\frac{r+s}{2}$ is rational.

H **19.** For all real numbers a and b, if $a < b$ then $a < \frac{a+b}{2} < b$. (You may use the properties of inequalities in T17–T27 of Appendix A.)

20. Given any two rational numbers r and s with $r < s$, there is another rational number between r and s. (*Hint:* Use the results of exercises 18 and 19.)

Use the properties of even and odd integers that are listed in Example 4.2.3 to do exercises 21–23. Indicate which properties you use to justify your reasoning.

21. True or false? If m is any even integer and n is any odd integer, then $m^2 + 3n$ is odd. Explain.

22. True or false? If a is any odd integer, then $a^2 + a$ is even. Explain.

23. True or false? If k is any even integer and m is any odd integer, then $(k+2)^2 - (m-1)^2$ is even. Explain.

Derive the statements in 24–26 as corollaries of Theorems 4.2.1, 4.2.2, and the results of exercises 12, 13, 14, 15, and 17.

24. For any rational numbers r and s, $2r + 3s$ is rational.

25. If r is any rational number, then $3r^2 - 2r + 4$ is rational.

26. For any rational number s, $5s^3 + 8s^2 - 7$ is rational.

27. It is a fact that if n is any nonnegative integer, then

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{1 - (1/2^{n+1})}{1 - (1/2)}.$$

(A more general form of this statement is proved in Section 5.2). Is the right-hand side of this equation rational? If so, express it as a ratio of two integers.

28. Suppose a, b, c, and d are integers and $a \neq c$. Suppose also that x is a real number that satisfies the equation

$$\frac{ax+b}{cx+d} = 1.$$

Must x be rational? If so, express x as a ratio of two integers.

✶ 29. Suppose a, b, and c are integers and x, y, and z are nonzero real numbers that satisfy the following equations:

$$\frac{xy}{x+y} = a \quad \text{and} \quad \frac{xz}{x+z} = b \quad \text{and} \quad \frac{yz}{y+z} = c.$$

Is x rational? If so, express it as a ratio of two integers.

30. Prove that if one solution for a quadratic equation of the form $x^2 + bx + c = 0$ is rational (where b and c are rational), then the other solution is also rational. (Use the fact that if the solutions of the equation are r and s, then $x^2 + bx + c = (x-r)(x-s)$.)

31. Prove that if a real number c satisfies a polynomial equation of the form

$$r_3x^3 + r_2x^2 + r_1x + r_0 = 0,$$

where r_0, r_1, r_2, and r_3 are rational numbers, then c satisfies an equation of the form

$$n_3x^3 + n_2x^2 + n_1x + n_0 = 0,$$

where n_0, n_1, n_2, and n_3 are integers.

Definition: A number c is called a **root** of a polynomial $p(x)$ if, and only if, $p(c) = 0$.

✶ 32. Prove that for all real numbers c, if c is a root of a polynomial with rational coefficients, then c is a root of a polynomial with integer coefficients.

Use the properties of even and odd integers that are listed in Example 4.2.3 to do exercises 33 and 34.

33. When expressions of the form $(x-r)(x-s)$ are multiplied out, a quadratic polynomial is obtained. For instance, $(x-2)(x-(-7)) = (x-2)(x+7) = x^2 + 5x - 14$.

H **a.** What can be said about the coefficients of the polynomial obtained by multiplying out $(x-r)(x-s)$ when both r and s are odd integers? when both r and s are even integers? when one of r and s is even and the other is odd?

b. It follows from part (a) that $x^2 - 1253x + 255$ cannot be written as a product of two polynomials with integer coefficients. Explain why this is so.

✶ 34. Observe that $(x-r)(x-s)(x-t)$

$$= x^3 - (r+s+t)x^2 + (rs+rt+st)x - rst.$$

a. Derive a result for cubic polynomials similar to the result in part (a) of exercise 33 for quadratic polynomials.

b. Can $x^3 + 7x^2 - 8x - 27$ be written as a product of three polynomials with integer coefficients? Explain.

In 35–39 find the mistakes in the "proofs" that the sum of any two rational numbers is a rational number.

35. "**Proof:** Any two rational numbers produce a rational number when added together. So if r and s are particular but arbitrarily chosen rational numbers, then $r+s$ is rational."

36. "**Proof:** Let rational numbers $r = \frac{1}{4}$ and $s = \frac{1}{2}$ be given. Then $r + s = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$, which is a rational number. This is what was to be shown."

37. "**Proof:** Suppose r and s are rational numbers. By definition of rational, $r = a/b$ for some integers a and b with $b \neq 0$, and $s = a/b$ for some integers a and b with $b \neq 0$. Then

$$r + s = \frac{a}{b} + \frac{a}{b} = \frac{2a}{b}.$$

Let $p = 2a$. Then p is an integer since it is a product of integers. Hence $r + s = p/b$, where p and b are integers and $b \neq 0$. Thus $r + s$ is a rational number by definition of rational. This is what was to be shown."

38. "**Proof:** Suppose r and s are rational numbers. Then $r = a/b$ and $s = c/d$ for some integers a, b, c, and d with $b \neq 0$ and $d \neq 0$ (by definition of rational). Then

$$r + s = \frac{a}{b} + \frac{c}{d}.$$

But this is a sum of two fractions, which is a fraction. So $r + s$ is a rational number since a rational number is a fraction."

39. "**Proof:** Suppose r and s are rational numbers. If $r + s$ is rational, then by definition of rational $r + s = a/b$ for some integers a and b with $b \neq 0$. Also since r and s are rational, $r = i/j$ and $s = m/n$ for some integers i, j, m, and n with $j \neq 0$ and $n \neq 0$. It follows that

$$r + s = \frac{i}{j} + \frac{m}{n} = \frac{a}{b},$$

which is a quotient of two integers with a nonzero denominator. Hence it is a rational number. This is what was to be shown."