Exercise Set 4.4

For each of the values of n and d given in 1–6, find integers q and r such that $n = dq + r$ and $0 \le r < d$.

1. $n = 70, d = 9$

2. $n = 62, d = 7$

3. $n = 36, d = 40$

4. $n = 3, d = 11$

5. $n = -45, d = 11$

6. $n = -27, d = 8$

Evaluate the expressions in 7–10.

7. a. 43 *div* 9 b. 43 *mod* 9

8. a. 50 *div* 7 b. 50 *mod* 7

9. a. 28 *div* 5 b. 28 *mod* 5

10. a. 30 *div* 2 b. 30 *mod* 2

11. Check the correctness of formula (4.4.1) given in Example 4.4.3 for the following values of *DayT* and N.
 a. $DayT = 6$ (Saturday) and $N = 15$
 b. $DayT = 0$ (Sunday) and $N = 7$
 c. $DayT = 4$ (Thursday) and $N = 12$

✶ 12. Justify formula (4.4.1) for general values of *DayT* and N.

13. On a Monday a friend says he will meet you again in 30 days. What day of the week will that be?

H **14.** If today is Tuesday, what day of the week will it be 1,000 days from today?

15. January 1, 2000, was a Saturday, and 2000 was a leap year. What day of the week will January 1, 2050, be?

16. Suppose d is a positive integer and n is any integer. If $d \mid n$, what is the remainder obtained when the quotient-remainder theorem is applied to n with divisor d?

17. Prove that the product of any two consecutive integers is even.

18. The result of exercise 17 suggests that the second apparent blind alley in the discussion of Example 4.4.7 might not be a blind alley after all. Write a new proof of Theorem 4.4.3 based on this observation.

19. Prove that for all integers n, $n^2 - n + 3$ is odd.

20. Suppose a is an integer. If $a \bmod 7 = 4$, what is $5a \bmod 7$? In other words, if division of a by 7 gives a remainder of 4, what is the remainder when $5a$ is divided by 7?

21. Suppose b is an integer. If $b \bmod 12 = 5$, what is $8b \bmod 12$? In other words, if division of b by 12 gives a remainder of 5, what is the remainder when $8b$ is divided by 12?

22. Suppose c is an integer. If $c \bmod 15 = 3$, what is $10c \bmod 15$? In other words, if division of c by 15 gives a remainder of 3, what is the remainder when $10c$ is divided by 15?

23. Prove that for all integers n, if $n \bmod 5 = 3$ then $n^2 \bmod 5 = 4$.

24. Prove that for all integers m and n, if $m \bmod 5 = 2$ and $n \bmod 3 = 6$ then $mn \bmod 5 = 1$.

25. Prove that for all integers a and b, if $a \bmod 7 = 5$ and $b \bmod 7 = 6$ then $ab \bmod 7 = 2$.

H **26.** Prove that a necessary and sufficient condition for a nonnegative integer n to be divisible by a positive integer d is that $n \bmod d = 0$.

27. Show that any integer n can be written in one of the three forms

$$n = 3q \quad \text{or} \quad n = 3q + 1 \quad \text{or} \quad n = 3q + 2$$

for some integer q.

28. a. Use the quotient-remainder theorem with $d = 3$ to prove that the product of any three consecutive integers is divisible by 3.

b. Use the *mod* notation to rewrite the result of part (a).

H **29.** a. Use the quotient-remainder theorem with $d = 3$ to prove that the square of any integer has the form $3k$ or $3k + 1$ for some integer k.

b. Use the *mod* notation to rewrite the result of part (a).

30. a. Use the quotient-remainder theorem with $d = 3$ to prove that the product of any two consecutive integers has the form $3k$ or $3k + 2$ for some integer k.

b. Use the *mod* notation to rewrite the result of part (a).

In 31–33, you may use the properties listed in Example 4.2.3.

31. a. Prove that for all integers m and n, $m + n$ and $m - n$ are either both odd or both even.

b. Find all solutions to the equation $m^2 - n^2 = 56$ for which both m and n are positive integers.

c. Find all solutions to the equation $m^2 - n^2 = 88$ for which both m and n are positive integers.

32. Given any integers a, b, and c, if $a - b$ is even and $b - c$ is even, what can you say about the parity of $2a - (b + c)$? Prove your answer.

33. Given any integers a, b, and c, if $a - b$ is odd and $b - c$ is even, what can you say about the parity of $a - c$? Prove your answer.

H **34.** Given any integer n, if $n > 3$, could n, $n + 2$, and $n + 4$ all be prime? Prove or give a counterexample.

Prove each of the statements in 35–46.

35. The fourth power of any integer has the form $8m$ or $8m + 1$ for some integer m.

H **36.** The product of any four consecutive integers is divisible by 8.

37. The square of any integer has the form $4k$ or $4k + 1$ for some integer k.

H **38.** For any integer n, $n^2 + 5$ is not divisible by 4.

H **39.** The sum of any four consecutive integers has the form $4k + 2$ for some integer k.

40. For any integer n, $n(n^2 - 1)(n + 2)$ is divisible by 4.

41. For all integers m, $m^2 = 5k$, or $m^2 = 5k + 1$, or $m^2 = 5k + 4$ for some integer k.

H **42.** Every prime number except 2 and 3 has the form $6q + 1$ or $6q + 5$ for some integer q.

43. If n is an odd integer, then n^4 *mod* $16 = 1$.

H **44.** For all real numbers x and y, $|x| \cdot |y| = |xy|$.

45. For all real numbers r and c with $c \geq 0$, if $-c \leq r \leq c$, then $|r| \leq c$.

46. For all real numbers r and c with $c \geq 0$, if $|r| \leq c$, then $-c \leq r \leq c$.

47. A matrix **M** has 3 rows and 4 columns.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

The 12 entries in the matrix are to be stored in *row major* form in locations 7,609 to 7,620 in a computer's memory. This means that the entries in the first row (reading left to right) are stored first, then the entries in the second row, and finally the entries in the third row.

a. Which location will a_{22} be stored in?

b. Write a formula (in i and j) that gives the integer n so that a_{ij} is stored in location $7{,}609 + n$.

c. Find formulas (in n) for r and s so that a_{rs} is stored in location $7{,}609 + n$.

48. Let **M** be a matrix with m rows and n columns, and suppose that the entries of **M** are stored in a computer's memory in row major form (see exercise 47) in locations $N, N + 1, N + 2, \ldots, N + mn - 1$. Find formulas in k for r and s so that a_{rs} is stored in location $N + k$.

✶ **49.** If m, n, and d are integers, $d > 0$, and m *mod* $d = n$ *mod* d, does it necessarily follow that $m = n$? That $m - n$ is divisible by d? Prove your answers.

✶ 50. If m, n, and d are integers, $d > 0$, and $d \mid (m - n)$, what is the relation between m *mod* d and n *mod* d? Prove your answer.

✶ 51. If m, n, a, b, and d are integers, $d > 0$, and m *mod* $d = a$ and n *mod* $d = b$, is $(m + n)$ *mod* $d = a + b$? Is $(m + n)$ *mod* $d = (a + b)$ *mod* d? Prove your answers.

✶ 52. If m, n, a, b, and d are integers, $d > 0$, and m *mod* $d = a$ and n *mod* $d = b$, is (mn) *mod* $d = ab$? Is (mn) *mod* $d = ab$ *mod* d? Prove your answers.

53. Prove that if m, d, and k are integers and $d > 0$, then $(m + dk)$ *mod* $d = m$ *mod* d.