Exercise Set 4.5

Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the values of x in 1–4.

1. 37.999 2. 17/4

3. −14.00001 4. −32/5

5. Use the floor notation to express 259 *div* 11 and 259 *mod* 11.

6. If k is an integer, what is $\lceil k \rceil$? Why?

7. If k is an integer, what is $\lceil k + \frac{1}{2} \rceil$? Why?

8. Seven pounds of raw material are needed to manufacture each unit of a certain product. Express the number of units that can be produced from n pounds of raw material using either the floor or the ceiling notation. Which notation is more appropriate?

9. Boxes, each capable of holding 36 units, are used to ship a product from the manufacturer to a wholesaler. Express the number of boxes that would be required to ship n units of the product using either the floor or the ceiling notation. Which notation is more appropriate?

10. If 0 = Sunday, 1 = Monday, 2 = Tuesday, ..., 6 = Saturday, then January 1 of year n occurs on the day of the week given by the following formula:

$$\left(n + \left\lfloor \frac{n-1}{4} \right\rfloor - \left\lfloor \frac{n-1}{100} \right\rfloor + \left\lfloor \frac{n-1}{400} \right\rfloor\right) \textit{mod } 7.$$

a. Use this formula to find January 1 of
i. 2050 ii. 2100 iii. the year of your birth.

H **b.** Interpret the different components of this formula.

11. State a necessary and sufficient condition for the floor of a real number to equal that number.

12. Prove that if n is any even integer, then $\lfloor n/2 \rfloor = n/2$.

13. Suppose n and d are integers and $d \neq 0$. Prove each of the following.
a. If $d \mid n$, then $n = \lfloor n/d \rfloor \cdot d$.
b. If $n = \lfloor n/d \rfloor \cdot d$, then $d \mid n$.
c. Use the floor notation to state a necessary and sufficient condition for an integer n to be divisible by an integer d.

Some of the statements in 14–22 are true and some are false. Prove each true statement and find a counterexample for each false statement, but do not use Theorem 4.5.1. in your proofs.

14. For all real numbers x and y, $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$.

15. For all real numbers x, $\lfloor x - 1 \rfloor = \lfloor x \rfloor - 1$.

16. For all real numbers x, $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.

H **17.** For all integers n,

$$\lfloor n/3 \rfloor = \begin{cases} n/3 & \text{if } n \textit{ mod } 3 = 0 \\ (n-1)/3 & \text{if } n \textit{ mod } 3 = 1 \\ (n-2)/3 & \text{if } n \textit{ mod } 3 = 2 \end{cases}.$$

H **18.** For all real numbers x and y, $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

H **19.** For all real numbers x, $\lceil x - 1 \rceil = \lceil x \rceil - 1$.

20. For all real numbers x and y, $\lceil xy \rceil = \lceil x \rceil \cdot \lceil y \rceil$.

21. For all odd integers n, $\lceil n/2 \rceil = (n+1)/2$.

22. For all real numbers x and y, $\lceil xy \rceil = \lceil x \rceil \cdot \lfloor y \rfloor$.

Prove each of the statements in 23–29.

23. For any real number x, if x is not an integer, then $\lfloor x \rfloor + \lfloor -x \rfloor = -1$.

24. For any integer m and any real number x, if x is not an integer, then $\lfloor x \rfloor + \lfloor m - x \rfloor = m - 1$.

H **25.** For all real numbers x, $\lfloor \lfloor x/2 \rfloor / 2 \rfloor = \lfloor x/4 \rfloor$.

26. For all real numbers x, if $x - \lfloor x \rfloor < 1/2$ then $\lfloor 2x \rfloor = 2\lfloor x \rfloor$.

27. For all real numbers x, if $x - \lfloor x \rfloor \geq 1/2$ then $\lfloor 2x \rfloor = 2\lfloor x \rfloor + 1$.

28. For any odd integer n,

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right).$$

29. For any odd integer n,

$$\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2+3}{4}.$$

30. Find the mistake in the following "proof" that $\lfloor n/2 \rfloor = (n-1)/2$ if n is an odd integer.
"**Proof:** Suppose n is any odd integer. Then $n = 2k + 1$ for some integer k. Consequently,

$$\left\lfloor \frac{2k+1}{2} \right\rfloor = \frac{(2k+1)-1}{2} = \frac{2k}{2} = k.$$

But $n = 2k + 1$. Solving for k gives $k = (n-1)/2$. Hence, by substitution, $\lfloor n/2 \rfloor = (n-1)/2$."