Exercise Set 4.7

1. A calculator display shows that $\sqrt{2} = 1.414213562$, and $1.414213562 = \frac{1414213562}{1000000000}$. This suggests that $\sqrt{2}$ is a rational number, which contradicts Theorem 4.7.1. Explain the discrepancy.

2. Example 4.2.1(h) illustrates a technique for showing that any repeating decimal number is rational. A calculator display shows the result of a certain calculation as 40.72727272727. Can you be sure that the result of the calculation is a rational number? Explain.

Determine which statements in 3–13 are true and which are false. Prove those that are true and disprove those that are false.

3. $6 - 7\sqrt{2}$ is irrational.

4. $3\sqrt{2} - 7$ is irrational.

5. $\sqrt{4}$ is irrational.

6. $\sqrt{2}/6$ is rational.

7. The sum of any two irrational numbers is irrational.

8. The difference of any two irrational numbers is irrational.

9. The positive square root of a positive irrational number is irrational.

10. If r is any rational number and s is any irrational number, then r/s is irrational.

11. The sum of any two positive irrational numbers is irrational.

12. The product of any two irrational numbers is irrational.

H **13.** If an integer greater than 1 is a perfect square, then its cube root is irrational.

14. Consider the following sentence: If x is rational then $\sqrt{x}$ is irrational. Is this sentence always true, sometimes true and sometimes false, or always false? Justify your answer.

15. a. Prove that for all integers a, if a^3 is even then a is even.
    b. Prove that $\sqrt[3]{2}$ is irrational.

16. a. Use proof by contradiction to show that for any integer n, it is impossible for n to equal both $3q_1 + r_1$ and $3q_2 + r_2$, where q_1, q_2, r_1, and r_2, are integers, $0 \le r_1 < 3$, $0 \le r_2 < 3$, and $r_1 \ne r_2$.
    b. Use proof by contradiction, the quotient-remainder theorem, division into cases, and the result of part (a) to prove that for all integers n, if n^2 is divisible by 3 then n is divisible by 3.
    c. Prove that $\sqrt{3}$ is irrational.

17. Give an example to show that if d is not prime and n^2 is divisible by d, then n need not be divisible by d.

H **18.** The quotient-remainder theorem says not only that there exist quotients and remainders but also that the quotient and remainder of a division are unique. Prove the uniqueness. That is, prove that if a and d are integers with $d > 0$ and if q_1, r_1, q_2, and r_2 are integers such that

$$a = dq_1 + r_1 \quad \text{where } 0 \le r_1 < d$$

and

$$a = dq_2 + r_2 \quad \text{where } 0 \le r_2 < d,$$

then

$$q_1 = q_2 \quad \text{and} \quad r_1 = r_2.$$

H **19.** Prove that $\sqrt{5}$ is irrational.

H **20.** Prove that for any integer a, $9 \nmid (a^2 - 3)$.

21. An alternative proof of the irrationality of $\sqrt{2}$ counts the number of 2's on the two sides of the equation $2n^2 = m^2$ and uses the unique factorization of integers theorem to deduce a contradiction. Write a proof that uses this approach.

22. Use the proof technique illustrated in exercise 21 to prove that if n is any integer that is not a perfect square, then $\sqrt{n}$ is irrational.

H **23.** Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

✶24. Prove that $\log_5(2)$ is irrational. (*Hint*: Use the unique factorisation of integers theorem.)

H **25.** Let $N = 2 \cdot 3 \cdot 5 \cdot 7 + 1$. What remainder is obtained when N is divided by 2? 3? 5? 7? Is N prime? Justify your answer.

H **26.** Suppose a is an integer and p is a prime number such that $p \mid a$ and $p \mid (a + 3)$. What can you deduce about p? Why?

27. Let $p_1, p_2, p_3, \ldots$ be a list of all prime numbers in ascending order. Here is a table of the first six:

p_1	p_2	p_3	p_4	p_5	p_6
2	3	5	7	11	13

H **a.** For each $i = 1, 2, 3, 4, 5, 6$, let $N_i = p_1 p_2 \cdots p_i + 1$. Calculate N_1, N_2, N_3, N_4, N_5, and N_6.

b. For each $i = 1, 2, 3, 4, 5, 6$, find the smallest prime number q_i such that q_i divides N_i. (*Hint*: Use the test for primality from exercise 31 in Section 4.6 to determine your answers.)

For exercises 28 and 29, use the fact that for all integers n,

$$n! = n(n-1)\ldots 3 \cdot 2 \cdot 1.$$

28. An alternative proof of the infinitude of the prime numbers begins as follows:

Proof: Suppose there are only finitely many prime numbers. Then one is the largest. Call it p. Let $M = p! + 1$. We will show that there is a prime number q such that $q > p$. Complete this proof.

H ✶**29.** Prove that for all integers n, if $n > 2$ then there is a prime number p such that $n < p < n!$.

H ✶**30.** Prove that if $p_1, p_2, \ldots$, and p_n are distinct prime numbers with $p_1 = 2$ and $n > 1$, then $p_1 p_2 \cdots p_n + 1$ can be written in the form $4k + 3$ for some integer k.

H **31.** a. Fermat's last theorem says that for all integers $n > 2$, the equation $x^n + y^n = z^n$ has no positive integer solution (solution for which x, y, and z are positive integers). Prove the following: If for all prime numbers $p > 2$, $x^p + y^p = z^p$ has no positive integer solution, then for any integer $n > 2$ that is not a power of 2, $x^n + y^n = z^n$ has no positive integer solution.

b. Fermat proved that there are no integers x, y, and z such that $x^4 + y^4 = z^4$. Use this result to remove the restriction in part (a) that n not be a power of 2. That is, prove that if n is a power of 2 and $n > 4$, then $x^n + y^n = z^n$ has no positive integer solution.

For exercises 32–35 note that to show there is a unique object with a certain property, show that (1) there is an object with the property and (2) if objects A and B have the property, then $A = B$.

32. Prove that there exists a unique prime number of the form $n^2 - 1$, where n is an integer that is greater than or equal to 2.

33. Prove that there exists a unique prime number of the form $n^2 + 2n - 3$, where n is a positive integer.

34. Prove that there is at most one real number a with the property that $a + r = r$ for all real numbers r. (Such a number is called an *additive identity*.)

35. Prove that there is at most one real number b with the property that $br = r$ for all real numbers r. (Such a number is called a *multiplicative identity*.)