Exercise Set 5.1*

Write the first four terms of the sequences defined by the formulas in 1–6.

1. $a_k = \dfrac{k}{10+k}$, for all integers $k \geq 1$.

2. $b_j = \dfrac{5-j}{5+j}$, for all integers $j \geq 1$.

3. $c_i = \dfrac{(-1)^i}{3^i}$, for all integers $i \geq 0$.

4. $d_m = 1 + \left(\dfrac{1}{2}\right)^m$ for all integers $m \geq 0$.

5. $e_n = \left\lfloor \dfrac{n}{2} \right\rfloor \cdot 2$, for all integers $n \geq 0$.

6. $f_n = \left\lfloor \dfrac{n}{4} \right\rfloor \cdot 4$, for all integers $n \geq 1$.

7. Let $a_k = 2k + 1$ and $b_k = (k-1)^3 + k + 2$ for all integers $k \geq 0$. Show that the first three terms of these sequences are identical but that their fourth terms differ.

Compute the first fifteen terms of each of the sequences in 8 and 9, and describe the general behavior of these sequences in words. (A definition of logarithm is given in Section 7.1.)

8. $g_n = \lfloor \log_2 n \rfloor$ for all integers $n \geq 1$.

9. $h_n = n \lfloor \log_2 n \rfloor$ for all integers $n \geq 1$.

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ✶ signals that an exercise is more challenging than usual.

Find explicit formulas for sequences of the form $a_1, a_2, a_3, \ldots$ with the initial terms given in 10–16.

10. $-1, 1, -1, 1, -1, 1$

11. $0, 1, -2, 3, -4, 5$

12. $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$

13. $1-\frac{1}{2}, \frac{1}{2}-\frac{1}{3}, \frac{1}{3}-\frac{1}{4}, \frac{1}{4}-\frac{1}{5}, \frac{1}{5}-\frac{1}{6}, \frac{1}{6}-\frac{1}{7}$

14. $\frac{1}{3}, \frac{4}{9}, \frac{9}{27}, \frac{16}{81}, \frac{25}{243}, \frac{36}{729}$

15. $0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}$

16. $3, 6, 12, 24, 48, 96$

✶ 17. Consider the sequence defined by $a_n = \frac{2n + (-1)^n - 1}{4}$ for all integers $n \geq 0$. Find an alternative explicit formula for a_n that uses the floor notation.

18. Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$, $a_4 = 0$, $a_5 = -1$, and $a_6 = -2$. Compute each of the summations and products below.

a. $\sum_{i=0}^{6} a_i$ **b.** $\sum_{i=0}^{0} a_i$ **c.** $\sum_{j=1}^{3} a_{2j}$ **d.** $\prod_{k=0}^{6} a_k$ e. $\prod_{k=2}^{2} a_k$

Compute the summations and products in 19–28.

19. $\sum_{k=1}^{5}(k+1)$ **20.** $\prod_{k=2}^{4} k^2$ 21. $\sum_{m=0}^{3} \frac{1}{2^m}$

22. $\prod_{j=0}^{4}(-1)^j$ **23.** $\sum_{i=1}^{1} i(i+1)$ 24. $\sum_{j=0}^{0}(j+1)\cdot 2^j$

25. $\prod_{k=2}^{2}\left(1-\frac{1}{k}\right)$ 26. $\sum_{k=-1}^{1}(k^2+3)$

27. $\sum_{n=1}^{10}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ 28. $\prod_{i=2}^{5}\frac{i(i+2)}{(i-1)\cdot(i+1)}$

Write the summations in 29–32 in expanded form.

29. $\sum_{i=1}^{n}(-2)^i$ 30. $\sum_{j=1}^{n} j(j+1)$ **31.** $\sum_{k=0}^{n+1}\frac{1}{k!}$ 32. $\sum_{i=1}^{k+1} i(i!)$

Evaluate the summations and products in 33–36 for the indicated values of the variable.

33. $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2};\ n=1$

34. $1(1!)+2(2!)+3(3!)+\ldots+m(m!);\ m=2$

35. $\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right)\cdots\left(\frac{k}{k+1}\right);\ k=3$

36. $\left(\frac{1\cdot 2}{3\cdot 4}\right)\left(\frac{4\cdot 5}{6\cdot 7}\right)\left(\frac{6\cdot 7}{8\cdot 9}\right)\cdots\left(\frac{m\cdot(m+1)}{(m+2)\cdot(m+3)}\right);\ m=1$

Rewrite 37–39 by separating off the final term.

37. $\sum_{i=1}^{k+1} i(i!)$ 38. $\sum_{k=1}^{m+1} k^2$ 39. $\sum_{m=1}^{n+1} m(m+1)$

Write each of 40–42 as a single summation.

40. $\sum_{i=1}^{k} i^3 + (k+1)^3$ 41. $\sum_{k=1}^{m}\frac{k}{k+1}+\frac{m+1}{m+2}$

42. $\sum_{m=0}^{n}(m+1)2^m + (n+2)2^{n+1}$

Write each of 43–52 using summation or product notation.

43. $1^2-2^2+3^2-4^2+5^2-6^2+7^2$

44. $(1^3-1)-(2^3-1)+(3^3-1)-(4^3-1)+(5^3-1)$

45. $(2^2-1)\cdot(3^2-1)\cdot(4^2-1)$

46. $\frac{2}{3\cdot 4}-\frac{3}{4\cdot 5}+\frac{4}{5\cdot 6}-\frac{5}{6\cdot 7}+\frac{6}{7\cdot 8}$

47. $1-r+r^2-r^3+r^4-r^5$

48. $(1-t)\cdot(1-t^2)\cdot(1-t^3)\cdot(1-t^4)$

49. $1^3+2^3+3^3+\cdots+n^3$

50. $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n}{(n+1)!}$

51. $n+(n-1)+(n-2)+\cdots+1$

52. $n+\frac{n-1}{2!}+\frac{n-2}{3!}+\frac{n-3}{4!}+\cdots+\frac{1}{n!}$

Transform each of 53 and 54 by making the change of variable $i = k+1$.

53. $\sum_{k=0}^{5} k(k-1)$ 54. $\prod_{k=1}^{n}\frac{k}{k^2+4}$

Transform each of 55–58 by making the change of variable $j = i - 1$.

55. $\sum_{i=1}^{n+1}\frac{(i-1)^2}{i\cdot n}$ **56.** $\sum_{i=3}^{n}\frac{i}{i+n-1}$

57. $\sum_{i=1}^{n-1}\frac{i}{(n-i)^2}$ 58. $\prod_{i=n}^{2n}\frac{n-i+1}{n+i}$

Write each of 59–61 as a single summation or product.

59. $3\cdot\sum_{k=1}^{n}(2k-3)+\sum_{k=1}^{n}(4-5k)$

60. $2\cdot\sum_{k=1}^{n}(3k^2+4)+5\cdot\sum_{k=1}^{n}(2k^2-1)$

61. $\left(\prod_{k=1}^{n}\frac{k}{k+1}\right)\cdot\left(\prod_{k=1}^{n}\frac{k+1}{k+2}\right)$

Compute each of 62–76. Assume the values of the variables are restricted so that the expressions are defined.

62. $\frac{4!}{3!}$ 63. $\frac{6!}{8!}$ 64. $\frac{4!}{0!}$

65. $\frac{n!}{(n-1)!}$ **66.** $\frac{(n-1)!}{(n+1)!}$ 67. $\frac{n!}{(n-2)!}$

68. $\dfrac{((n+1)!)^2}{(n!)^2}$ **69.** $\dfrac{n!}{(n-k)!}$ 70. $\dfrac{n!}{(n-k+1)!}$

71. $\binom{5}{3}$ 72. $\binom{7}{4}$ **73.** $\binom{3}{0}$

74. $\binom{5}{5}$ **75.** $\binom{n}{n-1}$ 76. $\binom{n+1}{n-1}$

77. **a.** Prove that $n! + 2$ is divisible by 2, for all integers $n \geq 2$.
b. Prove that $n! + k$ is divisible by k, for all integers $n \geq 2$ and $k = 2, 3, \ldots, n$.
H **c.** Given any integer $m \geq 2$, is it possible to find a sequence of $m - 1$ consecutive positive integers none of which is prime? Explain your answer.

78. Prove that for all nonnegative integers n and r with $r + 1 \leq n$, $\binom{n}{r+1} = \dfrac{n-r}{r+1}\binom{n}{r}$.

79. Prove that if p is a prime number and r is an integer with $0 < r < p$, then $\binom{p}{r}$ is divisible by p.

80. Suppose $a[1], a[2], a[3], \ldots, a[m]$ is a one-dimensional array and consider the following algorithm segment:

$sum := 0$
for $k := 1$ **to** m
  $sum := sum + a[k]$
next k

Fill in the blanks below so that each algorithm segment performs the same job as the one given previously.

a. $sum := 0$
for $i := 0$ **to** ______
  $sum :=$ ______
next i

b. $sum := 0$
for $j := 2$ **to** ______
  $sum :=$ ______
next j

Use repeated division by 2 to convert (by hand) the integers in 81–83 from base 10 to base 2.

81. 90 82. 98 83. 205

Make a trace table to trace the action of Algorithm 5.1.1 on the input in 84–86.

84. 23 85. 28 86. 44

87. Write an informal description of an algorithm (using repeated division by 16) to convert a nonnegative integer from decimal notation to hexadecimal notation (base 16).

Use the algorithm you developed for exercise 87 to convert the integers in 88–90 to hexadecimal notation.

88. 287 89. 693 90. 2,301

91. Write a formal version of the algorithm you developed for exercise 87.