Exercise Set 5.4

1. Suppose $a_1, a_2, a_3, \ldots$ is a sequence defined as follows:

$$a_1 = 1,\ a_2 = 3,$$
$$a_k = a_{k-2} + 2a_{k-1} \quad \text{for all integers } k \geq 3.$$

Prove that a_n is odd for all integers $n \geq 1$.

2. Suppose $b_1, b_2, b_3, \ldots$ is a sequence defined as follows:

$$b_1 = 4,\ b_2 = 12$$
$$b_k = b_{k-2} + b_{k-1} \quad \text{for all integers } k \geq 3.$$

Prove that b_n is divisible by 4 for all integers $n \geq 1$.

3. Suppose that $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$$c_0 = 2,\ c_1 = 2,\ c_2 = 6,$$
$$c_k = 3c_{k-3} \quad \text{for all integers } k \geq 3.$$

Prove that c_n is even for all integers $n \geq 0$.

4. Suppose that $d_1, d_2, d_3, \ldots$ is a sequence defined as follows:

$$d_1 = \frac{9}{10},\ d_2 = \frac{10}{11},$$
$$d_k = d_{k-1} \cdot d_{k-2} \quad \text{for all integers } k \geq 3.$$

Prove that $0 < d_n \leq 1$ for all integers $n \geq 0$.

5. Suppose that $e_0, e_1, e_2, \ldots$ is a sequence defined as follows:

$$e_0 = 12,\ e_1 = 29$$
$$e_k = 5e_{k-1} - 6e_{k-2} \quad \text{for all integers } k \geq 2.$$

Prove that $e_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 0$.

6. Suppose that $f_0, f_1, f_2, \ldots$ is a sequence defined as follows:

$$f_0 = 5,\ f_1 = 16$$
$$f_k = 7f_{k-1} - 10f_{k-2} \quad \text{for all integers } k \geq 2.$$

Prove that $f_n = 3 \cdot 2^n + 2 \cdot 5^n$ for all integers $n \geq 0$.

7. Suppose that $g_1, g_2, g_3, \ldots$ is a sequence defined as follows:

$$g_1 = 3,\ g_2 = 5$$
$$g_k = 3g_{k-1} - 2g_{k-2} \quad \text{for all integers } k \geq 3.$$

Prove that $g_n = 2^n + 1$ for all integers $n \geq 1$.

8. Suppose that $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:

$$h_0 = 1,\ h_1 = 2,\ h_2 = 3,$$
$$h_k = h_{k-1} + h_{k-2} + h_{k-3} \quad \text{for all integers } k \geq 3.$$

a. Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.

b. Suppose that s is any real number such that $s^3 \geq s^2 + s + 1$. (This implies that $s > 1.83$.) Prove that $h_n \leq s^n$ for all $n \geq 2$.

9. Define a sequence $a_1, a_2, a_3, \ldots$ as follows: $a_1 = 1, a_2 = 3$, and $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geq 3$. (This sequence is known as the Lucas sequence.) Use strong mathematical induction to prove that $a_n \leq \left(\frac{7}{4}\right)^n$ for all integers $n \geq 1$.

H 10. The problem that was used to introduce ordinary mathematical induction in Section 5.2 can also be solved using strong mathematical induction. Let $P(n)$ be "any collection of n coins can be obtained using a combination of 3¢ and 5¢ coins." Use strong mathematical induction to prove that $P(n)$ is true for all integers $n \geq 14$.

11. You begin solving a jigsaw puzzle by finding two pieces that match and fitting them together. Each subsequent step of the solution consists of fitting together two blocks made up of one or more pieces that have previously been assembled. Use strong mathematical induction to prove that the number of steps required to put together all n pieces of a jigsaw puzzle is $n - 1$.

H 12. The sides of a circular track contain a sequence of cans of gasoline. The total amount in the cans is sufficient to enable a certain car to make one complete circuit of the track, and it could all fit into the car's gas tank at one time. Use mathematical induction to prove that it is possible to find an initial location for placing the car so that it will be able to traverse the entire track by using the various amounts of gasoline in the cans that it encounters along the way.

H 13. Use strong mathematical induction to prove the existence part of the unique factorization of integers (Theorem 4.3.5): Every integer greater than 1 is either a prime number or a product of prime numbers.

14. Any product of two or more integers is a result of successive multiplications of two integers at a time. For instance,

here are a few of the ways in which $a_1a_2a_3a_4$ might be computed: $(a_1a_2)(a_3a_4)$ or $((a_1a_2)a_3)a_4$ or $a_1((a_2a_3)a_4)$. Use strong mathematical induction to prove that any product of two or more odd integers is odd.

15. Any sum of two or more integers is a result of successive additions of two integers at a time. For instance, here are a few of the ways in which $a_1 + a_2 + a_3 + a_4$ might be computed: $(a_1 + a_2) + (a_3 + a_4)$ or $((a_1 + a_2) + a_3) + a_4)$ or $a_1 + ((a_2 + a_3) + a_4)$. Use strong mathematical induction to prove that any sum of two or more even integers is even.

H **16.** Use strong mathematical induction to prove that for any integer $n \geq 2$, if n is even, then any sum of n odd integers is even, and if n is odd, then any sum of n odd integers is odd.

17. Compute $4^1, 4^2, 4^3, 4^4, 4^5, 4^6, 4^7$, and 4^8. Make a conjecture about the units digit of 4^n where n is a positive integer. Use strong mathematical induction to prove your conjecture.

18. Compute $9^0, 9^1, 9^2, 9^3, 9^4$, and 9^5. Make a conjecture about the units digit of 9^n where n is a positive integer. Use strong mathematical induction to prove your conjecture.

19. Find the mistake in the following "proof" that purports to show that every nonnegative integer power of every nonzero real number is 1.

"**Proof:** Let r be any nonzero real number and let the property $P(n)$ be the equation $r^n = 1$.

Show that P(0) is true: $P(0)$ is true because $r^0 = 1$ by definition of zeroth power.

Show that for all integers k ≥ 0, if P(i) is true for all integers i from 0 through k, then P(k + 1) is also true: Let k be any integer with $k \geq 0$ and suppose that $r^i = 1$ for all integers i from 0 through k. This is the inductive hypothesis. We must show that $r^{k+1} = 1$. Now

$$\begin{aligned} r^{k+1} &= r^{k+k-(k-1)} && \text{because } k + k - (k-1) = k + k - k + 1 = k + 1 \\ &= \frac{r^k \cdot r^k}{r^{k-1}} && \text{by the laws of exponents} \\ &= \frac{1 \cdot 1}{1} && \text{by inductive hypothesis} \\ &= 1. \end{aligned}$$

Thus $r^{k+1} = 1$ *[as was to be shown].*

[Since we have proved the basis step and the inductive step of the strong mathematical induction, we conclude that the given statement is true.]"

20. Use the well-ordering principle for the integers to prove Theorem 4.3.4: Every integer greater than 1 is divisible by a prime number.

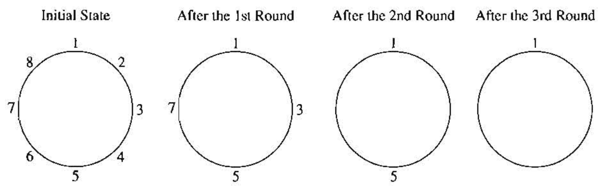
21. Use the well-ordering principle for the integers to prove the existence part of the unique factorization of integers theorem: Every integer greater than 1 is either prime or a product of prime numbers.

22. **a.** The Archimedean property for the rational numbers states that for all rational numbers r, there is an integer n such that $n > r$. Prove this property.
b. Prove that given any rational number r, the number $-r$ is also rational.
c. Use the results of parts (a) and (b) to prove that given any rational number r, there is an integer m such that $m < r$.

H **23.** Use the results of exercise 22 and the well-ordering principle for the integers to show that given any rational number r, there is an integer m such that $m \leq r < m + 1$.

24. Use the well-ordering principle to prove that given any integer $n \geq 1$, there exists an odd integer m and a nonnegative integer k such that $n = 2^k \cdot m$.

25. Imagine a situation in which eight people, numbered consecutively 1–8, are arranged in a circle. Starting from person #1, every second person in the circle is eliminated. The elimination process continues until only one person remains. In the first round the people numbered 2, 4, 6, and 8 are eliminated, in the second round the people numbered 3 and 7 are eliminated, and in the third round person #5 is eliminated. So after the third round only person #1 remains, as shown below.



a. Given a set of sixteen people arranged in a circle and numbered, consecutively 1–16, list the numbers of the people who are eliminated in each round if every second person is eliminated and the elimination process continues until only one person remains. Assume that the starting point is person #1.
b. Use mathematical induction to prove that for all integers $n \geq 1$, given any set of 2^n people arranged in a circle and numbered consecutively 1 through 2^n, if one starts from person #1 and goes repeatedly around the circle successively eliminating every second person, eventually only person #1 will remain.
c. Use the result of part (b) to prove that for any nonnegative integers n and m with $2^n \leq 2^n + m < 2^{n+1}$, if $r = 2^n + m$, then given any set of r people arranged in a circle and numbered consecutively 1 through r, if one starts from person #1 and goes repeatedly around the circle successively eliminating every second person, eventually only person #$(2m + 1)$ will remain.

26. Suppose $P(n)$ is a property such that
    1. $P(0)$, $P(1)$, $P(2)$ are all true,
    2. for all integers $k \geq 0$, if $P(k)$ is true, then $P(3k)$ is true.

    Must it follow that $P(n)$ is true for all integers $n \geq 0$? If yes, explain why; if no, give a counterexample.

27. Prove that if a statement can be proved by strong mathematical induction, then it can be proved by ordinary mathematical induction. To do this, let $P(n)$ be a property that is defined for integers n, and suppose the following two statements are true:
    1. $P(a)$, $P(a+1)$, $P(a+2)$, …, $P(b)$.
    2. For any integer $k \geq b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true.

    The principle of strong mathematical induction would allow us to conclude immediately that $P(n)$ is true for all integers $n \geq a$. Can we reach the same conclusion using the principle of ordinary mathematical induction? Yes! To see this, let $Q(n)$ be the property

    $$P(j) \text{ is true for all integers } j \text{ with } a \leq j \leq n.$$

    Then use ordinary mathematical induction to show that $Q(n)$ is true for all integers $n \geq b$. That is, prove
    1. $Q(b)$ is true.
    2. For any integer $k \geq b$, if $Q(k)$ is true then $Q(k+1)$ is true.

28. Give examples to illustrate the proof of Theorem 5.4.1.

H 29. It is a fact that every integer $n \geq 1$ can be written in the form

$$c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + \cdots + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0,$$

where $c_r = 1$ or 2 and $c_i = 0, 1,$ or 2 for all integers $i = 0, 1, 2, \ldots, r-1$. Sketch a proof of this fact.

H∗ 30. Use mathematical induction to prove the existence part of the quotient-remainder theorem for integers $n \geq 0$.

H∗ 31. Prove that if a statement can be proved by ordinary mathematical induction, then it can be proved by the well-ordering principle.

H 32. Use the principle of ordinary mathematical induction to prove the well-ordering principle for the integers.