Exercise Set 5.6

Find the first four terms of each of the recursively defined sequences in 1–8.

1. $a_k = 2a_{k-1} + k$, for all integers $k \geq 2$
$a_1 = 1$

2. $b_k = b_{k-1} + 3k$, for all integers $k \geq 2$
$b_1 = 1$

3. $c_k = k(c_{k-1})^2$, for all integers $k \geq 1$
$c_0 = 1$

4. $d_k = k(d_{k-1})^2$, for all integers $k \geq 1$
$d_0 = 3$

5. $s_k = s_{k-1} + 2s_{k-2}$, for all integers $k \geq 2$
$s_0 = 1, \ s_1 = 1$

6. $t_k = t_{k-1} + 2t_{k-2}$, for all integers $k \geq 2$
$t_0 = -1, \ t_1 = 2$

7. $u_k = ku_{k-1} - u_{k-2}$, for all integers $k \geq 3$
$u_1 = 1, \ u_2 = 1$

8. $v_k = v_{k-1} + v_{k-2} + 1$, for all integers $k \geq 3$
$v_1 = 1, \ v_2 = 3$

9. Let $a_0, a_1, a_2, \ldots$ be defined by the formula $a_n = 3n + 1$, for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $a_k = a_{k-1} + 3$, for all integers $k \geq 1$.

10. Let $b_0, b_1, b_2, \ldots$ be defined by the formula $b_n = 4^n$, for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $b_k = 4b_{k-1}$, for all integers $k \geq 1$.

11. Let $c_0, c_1, c_2, \ldots$ be defined by the formula $c_n = 2^n - 1$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation

$$c_k = 2c_{k-1} + 1.$$

12. Let $s_0, s_1, s_2, \ldots$ be defined by the formula $s_n = \dfrac{(-1)^n}{n!}$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation

$$s_k = \frac{-s_{k-1}}{k}.$$

13. Let $t_0, t_1, t_2, \ldots$ be defined by the formula $t_n = 2 + n$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation

$$t_k = 2t_{k-1} - t_{k-2}.$$

14. Let $d_0, d_1, d_2, \ldots$ be defined by the formula $d_n = 3^n - 2^n$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation

$$d_k = 5d_{k-1} - 6d_{k-2}.$$

H **15.** For the sequence of Catalan numbers defined in Example 5.6.4, prove that for all integers $n \geq 1$,

$$C_n = \frac{1}{4n+2}\binom{2n+2}{n+1}.$$

16. Use the recurrence relation and values for the Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ discussed in Example 5.6.5 to compute m_7 and m_8.

17. *Tower of Hanoi with Adjacency Requirement*: Suppose that in addition to the requirement that they never move a larger disk on top of a smaller one, the priests who move the disks of the Tower of Hanoi are also allowed only to move disks one by one from one pole to an *adjacent* pole. Assume poles A and C are at the two ends of the row and pole B is in the middle. Let

$$a_n = \begin{bmatrix} \text{the minimum number of moves} \\ \text{needed to transfer a tower of } n \\ \text{disks from pole } A \text{ to pole } C \end{bmatrix}.$$

a. Find a_1, a_2, and a_3. b. Find a_4.
c. Find a recurrence relation for $a_1, a_2, a_3, \ldots$.

18. *Tower of Hanoi with Adjacency Requirement*: Suppose the same situation as in exercise 17. Let

$$b_n = \begin{bmatrix} \text{the minimum number of moves} \\ \text{needed to transfer a tower of } n \\ \text{disks from pole } A \text{ to pole } B \end{bmatrix}.$$

a. Find b_1, b_2, and b_3. **b.** Find b_4.

c. Show that $b_k = a_{k-1} + 1 + b_{k-1}$ for all integers $k \geq 2$, where $a_1, a_2, a_3, \ldots$ is the sequence defined in exercise 17.

d. Show that $b_k \leq 3b_{k-1} + 1$ for all integers $k \geq 2$.

H ✶ **e.** Show that $b_k = 3b_{k-1} + 1$ for all integers $k \geq 2$.

19. *Four-Pole Tower of Hanoi*: Suppose that the Tower of Hanoi problem has four poles in a row instead of three. Disks can be transferred one by one from one pole to any other pole, but at no time may a larger disk be placed on top of a smaller disk. Let s_n be the minimum number of moves needed to transfer the entire tower of n disks from the left-most to the right-most pole.

a. Find s_1, s_2, and s_3. **b.** Find s_4.

c. Show that $s_k \leq 2s_{k-2} + 3$ for all integers $k \geq 3$.

20. *Tower of Hanoi Poles in a Circle:* Suppose that instead of being lined up in a row, the three poles for the original Tower of Hanoi are placed in a circle. The monks move the disks one by one from one pole to another, but they may only move disks one over in a clockwise direction and they may never move a larger disk on top of a smaller one. Let c_n be the minimum number of moves needed to transfer a pile of n disks from one pole to the next adjacent pole in the clockwise direction.

a. Justify the inequality $c_k \leq 4c_{k-1} + 1$ for all integers $k \geq 2$.

b. The expression $4c_{k-1} + 1$ is not the minimum number of moves needed to transfer a pile of k disks from one pole to another. Explain, for example, why $c_3 \neq 4c_2 + 1$.

21. *Double Tower of Hanoi*: In this variation of the Tower of Hanoi there are three poles in a row and $2n$ disks, two of each of n different sizes, where n is any positive integer. Initially one of the poles contains all the disks placed on top of each other in pairs of decreasing size. Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be placed on top of one of the same size. Let t_n be the minimum number of moves needed to transfer a tower of $2n$ disks from one pole to another.

a. Find t_1 and t_2. **b.** Find t_3.

c. Find a recurrence relation for $t_1, t_2, t_3, \ldots$.

22. *Fibonacci Variation*: A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions (which are more realistic than Fibonacci's):

(1) Rabbit pairs are not fertile during their first month of life but thereafter give birth to four new male/female pairs at the end of every month.

(2) No rabbits die.

a. Let r_n = the number of pairs of rabbits alive at the end of month n, for each integer $n \geq 1$, and let $r_0 = 1$. Find a recurrence relation for $r_0, r_1, r_2, \ldots$.

b. Compute $r_0, r_1, r_2, r_3, r_4, r_5$, and r_6.

c. How many rabbits will there be at the end of the year?

23. *Fibonacci Variation*: A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:

(1) Rabbit pairs are not fertile during their first *two* months of life, but thereafter give birth to three new male/female pairs at the end of every month.

(2) No rabbits die.

a. Let s_n = the number of pairs of rabbits alive at the end of month n, for each integer $n \geq 1$, and let $s_0 = 1$. Find a recurrence relation for $s_0, s_1, s_2, \ldots$.

b. Compute s_0, s_1, s_2, s_3, s_4, and s_5.

c. How many rabbits will there be at the end of the year?

In 24–34, $F_0, F_1, F_2, \ldots$ is the Fibonacci sequence.

24. Use the recurrence relation and values for $F_0, F_1, F_2, \ldots$ given in Example 5.6.6 to compute F_{13} and F_{14}.

25. The Fibonacci sequence satisfies the recurrence relation $F_k = F_{k-1} + F_{k-2}$, for all integers $k \geq 2$.

a. Explain why the following is true:

$$F_{k+1} = F_k + F_{k-1} \quad \text{for all integers } k \geq 1.$$

b. Write an equation expressing F_{k+2} in terms of F_{k+1} and F_k.

c. Write an equation expressing F_{k+3} in terms of F_{k+2} and F_{k+1}

26. Prove that $F_k = 3F_{k-3} + 2F_{k-4}$ for all integers $k \geq 4$.

27. Prove that $F_k^2 - F_{k-1}^2 = F_kF_{k-1} - F_{k+1}F_{k-1}$, for all integers $k \geq 1$.

28. Prove that $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_kF_{k-1}$, for all integers $k \geq 1$.

29. Prove that $F_{k+1}^2 - F_k^2 = F_{k-1}F_{k+2}$, for all integers $k \geq 1$.

30. Use mathematical induction to prove that for all integers $n \geq 0$, $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$.

✶ 31. Use strong mathematical induction to prove that $F_n < 2^n$ for all integers $n \geq 1$.

H ✶ **32.** Let $F_0, F_1, F_2, \ldots$ be the Fibonacci sequence defined in Section 5.6. Prove that for all integers $n \geq 0$, $\gcd(F_{n+1}, F_n) = 1$.

33. It turns out that the Fibonacci sequence satisfies the following explicit formula: For all integers $F_n \geq 0$,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Verify that the sequence defined by this formula satisfies the recurrence relation $F_k = F_{k-1} + F_{k-2}$ for all integers $k \geq 2$.

H **34.** (For students who have studied calculus) Find $\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)$, assuming that the limit exists.

H ✶ **35.** (For students who have studied calculus) Prove that $\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)$ exists.

36. (For students who have studied calculus) Define $x_0, x_1, x_2, \ldots$ as follows:

$$x_k = \sqrt{2 + x_{k-1}} \quad \text{for all integers } k \geq 1$$
$$x_0 = 0$$

Find $\lim_{n\to\infty} x_n$. (Assume that the limit exists.)

37. *Compound Interest*: Suppose a certain amount of money is deposited in an account paying 4% annual interest compounded quarterly. For each positive integer n, let R_n = the amount on deposit at the end of the nth quarter, assuming no additional deposits or withdrawals, and let R_0 be the initial amount deposited.

a. Find a recurrence relation for $R_0, R_1, R_2, \ldots$.

b. If $R_0 = \$5000$, find the amount of money on deposit at the end of one year.

c. Find the APR for the account.

38. *Compound Interest*: Suppose a certain amount of money is deposited in an account paying 3% annual interest compounded monthly. For each positive integer n, let S_n = the amount on deposit at the end of the nth month, and let S_0 be the initial amount deposited.

a. Find a recurrence relation for $S_0, S_1, S_2, \ldots$, assuming no additional deposits or withdrawals during the year.

b. If $S_0 = \$10,000$, find the amount of money on deposit at the end of one year.

c. Find the APR for the account.

39. With each step you take when climbing a staircase, you can move up either one stair or two stairs. As a result, you can climb the entire staircase taking one stair at a time, taking two at a time, or taking a combination of one- and two-stair increments. For each integer $n \geq 1$, if the staircase consists of n stairs, let c_n be the number of different ways to climb the staircase. Find a recurrence relation for $c_1, c_2, c_3, \ldots$.

40. A set of blocks contains blocks of heights 1, 2, and 4 centimeters. Imagine constructing towers by piling blocks of different heights directly on top of one another. (A tower of height 6 cm could be obtained using six 1-cm blocks, three 2-cm blocks one 2-cm block with one 4-cm block on top, one 4-cm block with one 2-cm block on top, and so forth.) Let t be the number of ways to construct a tower of height n cm using blocks from the set. (Assume an unlimited supply of blocks of each size.) Find a recurrence relation for $t_1, t_2, t_3, \ldots$.

41. Use the recursive definition of summation, together with mathematical induction, to prove the generalized distributive law that for all positive integers n, if $a_1, a_2, \ldots, a_n$ and c are real numbers, then

$$\sum_{i=1}^{n} ca_i = c\left(\sum_{i=1}^{n} a_i\right).$$

42. Use the recursive definition of product, together with mathematical induction, to prove that for all positive integers n, if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$\prod_{i=1}^{n}(a_i b_i) = \left(\prod_{i=1}^{n} a_i\right)\left(\prod_{i=1}^{n} b_i\right).$$

43. Use the recursive definition of product, together with mathematical induction, to prove that for all positive integers n, if $a_1, a_2, \ldots, a_n$ and c are real numbers, then

$$\prod_{i=1}^{n}(ca_i) = c^n\left(\prod_{i=1}^{n} a_i\right).$$

H **44.** The triangle inequality for absolute value states that for all real numbers a and b, $|a + b| \leq |a| + |b|$. Use the recursive definition of summation, the triangle inequality, the definition of absolute value, and mathematical induction to prove that for all positive integers n, if $a_1, a_2, \ldots, a_n$ are real numbers, then

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|.$$