Exercise Set 5.7

1. The formula

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

is true for all integers $n \geq 1$. Use this fact to solve each of the following problems:

a. If k is an integer and $k \geq 2$, find a formula for the expression $1+2+3+\cdots+(k-1)$.

b. If n is an integer and $n \geq 1$, find a formula for the expression $3+2+4+6+8+\cdots+2n$.

c. If n is an integer and $n \geq 1$, find a formula for the expression $3+3\cdot 2+3\cdot 3+\cdots+3\cdot n+n$.

2. The formula

$$1+r+r^2+\cdots+r^n=\frac{r^{n+1}-1}{r-1}$$

is true for all real numbers r except $r = 1$ and for all integers $n \geq 0$. Use this fact to solve each of the following problems:

a. If i is an integer and $i \geq 1$, find a formula for the expression $1 + 2 + 2^2 + \cdots + 2^{i-1}$.

b. If n is an integer and $n \geq 1$, find a formula for the expression $3^{n-1} + 3^{n-2} + \cdots + 3^2 + 3 + 1$.

c. If n is an integer and $n \geq 2$, find a formula for the expression $2^n + 2^{n-2} \cdot 3 + 2^{n-3} \cdot 3 + \cdots + 2^2 \cdot 3 + 2 \cdot 3 + 3$

d. If n is an integer and $n \geq 1$, find a formula for the expression

$$2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \cdots + (-1)^{n-1} \cdot 2 + (-1)^n.$$

In each of 3–15 a sequence is defined recursively. Use iteration to guess an explicit formula for the sequence. Use the formulas from Section 5.2 to simplify your answers whenever possible.

3. $a_k = ka_{k-1}$, for all integers $k \geq 1$
$a_0 = 1$

4. $b_k = \dfrac{b_{k-1}}{1 + b_{k-1}}$, for all integers $k \geq 1$
$b_0 = 1$

5. $c_k = 3c_{k-1} + 1$, for all integers $k \geq 2$
$c_1 = 1$

***H* 6.** $d_k = 2d_{k-1} + 3$, for all integers $k \geq 2$
$d_1 = 2$

7. $e_k = 4e_{k-1} + 5$, for all integers $k \geq 1$
$e_0 = 2$

8. $f_k = f_{k-1} + 2^k$, for all integers $k \geq 2$
$f_1 = 1$

***H* 9.** $g_k = \dfrac{g_{k-1}}{g_{k-1} + 2}$, for all integers $k \geq 2$
$g_1 = 1$

10. $h_k = 2^k - h_{k-1}$, for all integers $k \geq 1$
$h_0 = 1$

11. $p_k = p_{k-1} + 2 \cdot 3^k$
$p_1 = 2$

12. $s_k = s_{k-1} + 2k$, for all integers $k \geq 1$
$s_0 = 3$

13. $t_k = t_{k-1} + 3k + 1$, for all integers $k \geq 1$
$t_0 = 0$

✶ 14. $x_k = 3x_{k-1} + k$, for all integers $k \geq 2$
$x_1 = 1$

15. $y_k = y_{k-1} + k^2$, for all integers $k \geq 2$
$y_1 = 1$

16. Solve the recurrence relation obtained as the answer to exercise 18(c) of Section 5.6.

17. Solve the recurrence relation obtained as the answer to exercise 21(c) of Section 5.6.

18. Suppose d is a fixed constant and $a_0, a_1, a_2, \ldots$ is a sequence that satisfies the recurrence relation $a_k = a_{k-1} + d$, for all integers $k \geq 1$. Use mathematical induction to prove that $a_n = a_0 + nd$, for all integers $n \geq 0$.

19. A worker is promised a bonus if he can increase his productivity by 2 units a day every day for a period of 30 days. If on day 0 he produces 170 units, how many units must he produce on day 30 to qualify for the bonus?

20. A runner targets herself to improve her time on a certain course by 3 seconds a day. If on day 0 she runs the course in 3 minutes, how fast must she run it on day 14 to stay on target?

21. Suppose r is a fixed constant and $a_0, a_1, a_2 \ldots$ is a sequence that satisfies the recurrence relation $a_k = ra_{k-1}$, for all integers $k \geq 1$ and $a_0 = a$. Use mathematical induction to prove that $a_n = ar^n$, for all integers $n \geq 0$.

22. As shown in Example 5.6.8, if a bank pays interest at a rate of i compounded m times a year, then the amount of money P_k at the end of k time periods (where one time period $= 1/m$th of a year) satisfies the recurrence relation $P_k = [1 + (i/m)]P_{k-1}$ with initial condition $P_0 =$ the initial amount deposited. Find an explicit formula for P_n.

23. Suppose the population of a country increases at a steady rate of 3% per year. If the population is 50 million at a certain time, what will it be 25 years later?

24. A chain letter works as follows: One person sends a copy of the letter to five friends, each of whom sends a copy to five friends, each of whom sends a copy to five friends, and so forth. How many people will have received copies of the letter after the twentieth repetition of this process, assuming no person receives more than one copy?

25. A certain computer algorithm executes twice as many operations when it is run with an input of size k as when it is run with an input of size $k - 1$ (where k is an integer that is greater than 1). When the algorithm is run with an input of size 1, it executes seven operations. How many operations does it execute when it is run with an input of size 25?

26. A person saving for retirement makes an initial deposit of \$1,000 to a bank account earning interest at a rate of 3% per year compounded monthly, and each month she adds an additional \$200 to the account.

a. For each nonnegative integer n, let A_n be the amount in the account at the end of n months. Find a recurrence relation relating A_k to A_{k-1}.

***H* b.** Use iteration to find an explicit formula for A_n.

c. Use mathematical induction to prove the correctness of the formula you obtained in part (b).

d. How much will the account be worth at the end of 20 years? At the end of 40 years?

***H* e.** In how many years will the account be worth \$10,000?

27. A person borrows \$3,000 on a bank credit card at a nominal rate of 18% per year, which is actually charged at a rate of 1.5% per month.
   H a. What is the annual percentage rate (APR) for the card? (See Example 5.6.8 for a definition of APR.)
   b. Assume that the person does not place any additional charges on the card and pays the bank \$150 each month to pay off the loan. Let B_n be the balance owed on the card after n months. Find an explicit formula for B_n.
   H c. How long will be required to pay off the debt?
   d. What is the total amount of money the person will have paid for the loan?

In 28–42 use mathematical induction to verify the correctness of the formula you obtained in the referenced exercise.

28. Exercise 3	29. Exercise 4	30. Exercise 5
31. Exercise 6	32. Exercise 7	33. Exercise 8
34. Exercise 9	**H** 35. Exercise 10	36. Exercise 11
H 37. Exercise 12	38. Exercise 13	39. Exercise 14
40. Exercise 15	41. Exercise 16	42. Exercise 17

In each of 43–49 a sequence is defined recursively. (a) Use iteration to guess an explicit formula for the sequence. (b) Use strong mathematical induction to verify that the formula of part (a) is correct.

43. $a_k = \dfrac{a_{k-1}}{2a_{k-1} - 1}$, for all integers $k \geq 1$
$a_0 = 2$

44. $b_k = \dfrac{2}{b_{k-1}}$, for all integers $k \geq 2$
$b_1 = 1$

45. $v_k = v_{\lfloor k/2 \rfloor} + v_{\lfloor (k+1)/2 \rfloor} + 2$, for all integers $k \geq 2$,
$v_1 = 1$.

H 46. $s_k = 2s_{k-2}$, for all integers $k \geq 2$,
$s_0 = 1, s_1 = 2$.

47. $t_k = k - t_{k-1}$, for all integers $k \geq 1$,
$t_0 = 0$.

H 48. $w_k = w_{k-2} + k$, for all integers $k \geq 3$,
$w_1 = 1, w_2 = 2$.

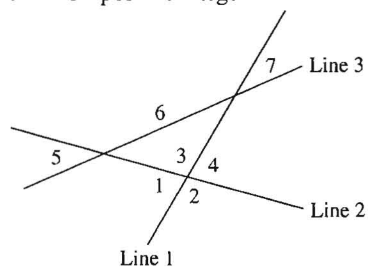
H 49. $u_k = u_{k-2} \cdot u_{k-1}$, for all integers $k \geq 2$,
$u_0 = u_1 = 2$.

In 50 and 51 determine whether the given recursively defined sequence satisfies the explicit formula $a_n = (n-1)^2$, for all integers $n \geq 1$.

50. $a_k = 2a_{k-1} + k - 1$, for all integers $k \geq 2$
$a_1 = 0$

51. $a_k = (a_{k-1} + 1)^2$, for all integers $k \geq 2$
$a_1 = 0$

52. A single line divides a plane into two regions. Two lines (by crossing) can divide a plane into four regions; three lines can divide it into seven regions (see the figure). Let P_n be the maximum number of regions into which n lines divide a plane, where n is a positive integer.



   a. Derive a recurrence relation for P_k in terms of P_{k-1}, for all integers $k \geq 2$.
   b. Use iteration to guess an explicit formula for P_n.

53. Compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ for small values of n (up to about 5 or 6). Conjecture explicit formulas for the entries in this matrix, and prove your conjecture using mathematical induction.

54. In economics the behavior of an economy from one period to another is often modeled by recurrence relations. Let Y_k be the income in period k and C_k be the consumption in period k. In one economic model, income in any period is assumed to be the sum of consumption in that period plus investment and government expenditures (which are assumed to be constant from period to period), and consumption in each period is assumed to be a linear function of the income of the preceding period. That is,

$Y_k = C_k + E$ where E is the sum of investment plus government expenditures

$C_k = c + mY_{k-1}$ where c and m are constants.

Substituting the second equation into the first gives $Y_k = E + c + mY_{k-1}$.
   a. Use iteration on the above recurrence relation to obtain

$$Y_n = (E + c)\left(\frac{m^n - 1}{m - 1}\right) + m^n Y_0$$

   for all integers $n \geq 1$.
   b. (For students who have studied calculus) Show that if $0 < m < 1$, then $\lim_{n \to \infty} Y_n = \dfrac{E + c}{1 - m}$.