Exercise Set 5.8

1. Which of the following are second-order linear homogeneous recurrence relations with constant coefficients?
   a. $a_k = 2a_{k-1} - 5a_{k-2}$
   b. $b_k = kb_{k-1} + b_{k-2}$
   c. $c_k = 3c_{k-1} \cdot c_{k-2}^2$
   d. $d_k = 3d_{k-1} + d_{k-2}$
   e. $r_k = r_{k-1} - r_{k-2} - 2$
   f. $s_k = 10s_{k-2}$

2. Which of the following are second-order linear homogeneous recurrence relations with constant coefficients?
   a. $a_k = (k-1)a_{k-1} + 2ka_{k-2}$
   b. $b_k = -b_{k-1} + 7b_{k-2}$
   c. $c_k = 3c_{k-1} + 1$
   d. $d_k = 3d_{k-1}^2 + d_{k-2}$
   e. $r_k = r_{k-1} - 6r_{k-3}$
   f. $s_k = s_{k-1} + 10s_{k-2}$

3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the explicit formula

$$a_n = C \cdot 2^n + D \quad \text{for all integers } n \geq 0,$$

where C and D are real numbers.

   a. Find C and D so that $a_0 = 1$ and $a_1 = 3$. What is a_2 in this case?
   b. Find C and D so that $a_0 = 0$ and $a_1 = 2$. What is a_2 in this case?

4. Let $b_0, b_1, b_2, \ldots$ be the sequence defined by the explicit formula
$$b_n = C \cdot 3^n + D(-2)^n \quad \text{for all integers } n \geq 0,$$
where C and D are real numbers.
   a. Find C and D so that $b_0 = 0$ and $b_1 = 5$. What is b_2 in this case?
   b. Find C and D so that $b_0 = 3$ and $b_1 = 4$. What is b_2 in this case?

5. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the explicit formula
$$a_n = C \cdot 2^n + D \quad \text{for all integers } n \geq 0,$$
where C and D are real numbers. Show that for any choice of C and D,
$$a_k = 3a_{k-1} - 2a_{k-2} \quad \text{for all integers } k \geq 2.$$

6. Let $b_0, b_1, b_2, \ldots$ be the sequence defined by the explicit formula
$$b_n = C \cdot 3^n + D(-2)^n \quad \text{for all integers } n \geq 0,$$
where C and D are real numbers. Show that for any choice of C and D,
$$b_k = b_{k-1} + 6b_{k-2} \quad \text{for all integers } k \geq 2.$$

7. Solve the system of equations in Example 5.8.4 to obtain
$$C = \frac{1+\sqrt{5}}{2\sqrt{5}} \text{ and } D = \frac{-(1-\sqrt{5})}{2\sqrt{5}}.$$

In each of 8–10: (a) suppose a sequence of the form $1, t, t^2, t^3 \ldots t^n \ldots$ where $t \neq 0$, satisfies the given recurrence relation (but not necessarily the initial conditions), and find all possible values of t: (b) suppose a sequence satisfies the given initial conditions as well as the recurrence relation, and find an explicit formula for the sequence.

8. $a_k = 2a_{k-1} + 3a_{k-2}$, for all integers $k \geq 2$
$a_0 = 1, a_1 = 2$

9. $b_k = 7b_{k-1} - 10b_{k-2}$, for all integers $k \geq 2$
$b_0 = 2, b_1 = 2$

10. $c_k = c_{k-1} + 6c_{k-2}$, for all integers $k \geq 2$
$c_0 = 0, c_1 = 3$

In each of 11–16 suppose a sequence satisfies the given recurrence relation and initial conditions. Find an explicit formula for the sequence.

11. $d_k = 4d_{k-2}$, for all integers $k \geq 2$
$d_0 = 1,\ d_1 = -1$

12. $e_k = 9e_{k-2}$, for all integers $k \geq 2$
$e_0 = 0,\ e_1 = 2$

13. $r_k = 2r_{k-1} - r_{k-2}$, for all integers $k \geq 2$
$r_0 = 1,\ r_1 = 4$

14. $s_k = -4s_{k-1} - 4s_{k-2}$, for all integers $k \geq 2$
$s_0 = 0,\ s_1 = -1$

15. $t_k = 6t_{k-1} - 9t_{k-2}$, for all integers $k \geq 2$
$t_0 = 1,\ t_1 = 3$

H **16.** $s_k = 2s_{k-1} + 2s_{k-2}$, for all integers $k \geq 2$
$s_0 = 1,\ s_1 = 3$

17. Find an explicit formula for the sequence of exercise 39 in Section 5.6

18. Suppose that the sequences $s_0, s_1, s_2, \ldots$ and $t_0, t_1, t_2, \ldots$ both satisfy the same second-order linear homogeneous recurrence relation with constant coefficients:
$$s_k = 5s_{k-1} - 4s_{k-2} \quad \text{for all integers } k \geq 2,$$
$$t_k = 5t_{k-1} - 4t_{k-2} \quad \text{for all integers } k \geq 2.$$
Show that the sequence $2s_0 + 3t_0, 2s_1 + 3t_1, 2s_2 + 3t_2, \ldots$ also satisfies the same relation. In other words, show that
$$2s_k + 3t_k = 5(2s_{k-1} + 3t_{k-1}) - 4(2s_{k-2} + 3t_{k-2})$$
for all integers $k \geq 2$. Do *not* use Lemma 5.8.2.

19. Show that if r, s, a_0, and a_1 are numbers with $r \neq s$, then there exist unique numbers C and D so that
$$C + D = a_0$$
$$Cr + Ds = a_1.$$

20. Show that if r is a nonzero real number, k and m are distinct integers, and a_k and a_m are any real numbers, then there exist unique real numbers C and D so that
$$Cr^k + kDr^k = a_k$$
$$Cr^m + lDr^m = a_m.$$

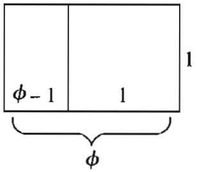
H **21.** Prove Theorem 5.8.5 for the case where the values of C and D are determined by a_0 and a_1.

Exercises 22 and 23 are intended for students who are familiar with complex numbers.

22. Find an explicit formula for a sequence $a_0, a_1, a_2, \ldots$ that satisfies
$$a_k = 2a_{k-1} - 2a_{k-2} \quad \text{for all integers } k \geq 2$$
with initial conditions $a_0 = 1$ and $a_1 = 2$.

23. Find an explicit formula for a sequence $b_0, b_1, b_2, \ldots$ that satisfies
$$b_k = 2b_{k-1} - 5b_{k-2} \quad \text{for all integers } k \geq 2$$
with initial conditions $b_0 = 1$ and $b_1 = 1$.

24. The numbers $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ that appear in the explicit formula for the Fibonacci sequence are related to a quantity called the *golden ratio* in Greek mathematics. Consider a rectangle of length ϕ units and height 1, where $\phi > 1$.



Divide the rectangle into a rectangle and a square as shown in the preceding diagram. The square is 1 unit on each side, and the rectangle has sides of lengths 1 and $\phi - 1$.

The ancient Greeks considered the outer rectangle to be perfectly proportioned (saying that the lengths of its sides were in a *golden ratio* to each other) if the ratio of the length to the width of the outer rectangle equaled the ratio of the length to the width of the inner rectangle. That is,

$$\frac{\phi}{1} = \frac{1}{\phi - 1}.$$

a. Show that ϕ satisfies the following quadratic equation: $t^2 - t - 1 = 0$.

b. Find the two solutions of $t^2 - t - 1 = 0$ and call them ϕ_1 and ϕ_2.

c. Express the explicit formula for the Fibonacci sequence in terms of ϕ_1 and ϕ_2.