Exercise Set 5.9

1. Consider the set of Boolean expressions defined in Example 5.9.1. Give derivations showing that each of the following is a Boolean expression over the English alphabet $\{a, b, c, \ldots, x, y, z\}$.
   a. $(\sim p \vee (q \wedge (r \vee \sim s)))$
   b. $((p \vee q) \vee \sim((p \wedge \sim s) \wedge r))$

2. Let S be defined as in Example 5.9.2. Give derivations showing that each of the following is in S.
   a. aab b. bb

3. Consider the MIU-system discussed in Example 5.9.3. Give derivations showing that each of the following is in the MIU-system.
   a. $MIUI$
   b. $MUIIU$

4. The set of arithmetic expressions over the real numbers can be defined recursively as follows:
   I. BASE: Each real number r is an arithmetic expression.
   II. RECURSION: If u and v are arithmetic expressions, then the following are also arithmetic expressions:
   a. $(+u)$ b. $(-u)$
   c. $(u+v)$ d. $(u-v)$
   e. $(u \cdot v)$ f. $\left(\frac{u}{v}\right)$
   III. RESTRICTION: There are no arithmetic expressions over the real numbers other than those obtained from I and II.

   (Note that the *expression* $\left(\frac{u}{v}\right)$ is legal even though the value of v may be 0.) Give derivations showing that each of the following is an arithmetic expression.

   a. $((2 \cdot (0.3\text{–}4.2)) + (-7))$ b. $\left(\frac{(9 \cdot (6.1 + 2))}{((4\text{–}7) \cdot 6)}\right)$

5. Define a set S recursively as follows:
   I. BASE: $1 \in S$
   II. RECURSION: If $s \in S$, then
   a. $0s \in S$ b. $1s \in S$
   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every string in S ends in a 1.

6. Define a set S recursively as follows:
   I. BASE: $a \in S$
   II. RECURSION: If $s \in S$, then,
   a. $sa \in S$ b. $sb \in S$
   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every string in S begins with an a.

7. Define a set S recursively as follows:
   I. BASE: $\epsilon \in S$
   II. RECURSION: If $s \in S$, then
   a. $bs \in S$ b. $sb \in S$
   c. $saa \in S$ d. $aas \in S$
   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every string in S contains an even number of a's.

8. Define a set S recursively as follows:
   I. BASE: $1 \in S$, $2 \in S$, $3 \in S$, $4 \in S$, $5 \in S$, $6 \in S$, $7 \in S$, $8 \in S$, $9 \in S$
   II. RECURSION: If $s \in S$ and $t \in S$, then
   a. $s0 \in S$ b. $st \in S$
   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that no string in S represents an integer with a leading zero.

H 9. Define a set S recursively as follows:
   I. BASE: $1 \in S$, $3 \in S$, $5 \in S$, $7 \in S$, $9 \in S$
   II. RECURSION: If $s \in S$ and $t \in S$ then
   a. $st \in S$ b. $2s \in S$
   c. $4s \in S$ d. $6s \in S$
   e. $8s \in S$
   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every string in S represents an odd integer.

H 10. Define a set S recursively as follows:
   I. BASE: $0 \in S$, $5 \in S$
   II. RECURSION: If $s \in S$ and $t \in S$ then
   a. $s + t \in S$ b. $s - t \in S$
   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every integer in S is divisible by 5.

11. Define a set S recursively as follows:
   I. BASE: $0 \in S$
   II. RECURSION: If $s \in S$, then
   a. $s + 3 \in S$ b. $s - 3 \in S$
   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every integer in S is divisible by 3.

H✱ 12. Is the string MU in the MIU-system? Use structural induction to prove your answer.

13. Consider the set P of parenthesis structures defined in Example 5.9.4. Give derivations showing that each of the following is in P.
   a. ()(()) b. (())(())

✱ 14. Determine whether either of the following parenthesis structures is in the set P defined in Example 5.9.4. Use structural induction to prove your answers.
   a. ()(() b. (()()))(()

15. Give a recursive definition for the set of all strings of 0's and 1's that have the same number of 0's as 1's.

16. Give a recursive definition for the set of all strings of 0's and 1's for which all the 0's precede all the 1's.

17. Give a recursive definition for the set of all strings of a's and b's that contain an odd number of a's.

18. Give a recursive definition for the set of all strings of a's and b's that contain exactly one a.

19. Use the definition of McCarthy's 91 function in Example 5.9.6 to show the following:
   a. $M(86) = M(91)$ b. $M(91) = 91$

✱ 20. Prove that McCarthy's 91 function equals 91 for all positive integers less than or equal to 101.

21. Use the definition of the Ackermann function in Example 5.9.7 to compute the following:
   a. $A(1, 1)$ b. $A(2, 1)$

22. Use the definition of the Ackermann function to show the following:
   a. $A(1, n) = n + 2$, for all nonnegative integers n.
   b. $A(2, n) = 3 + 2n$, for all nonnegative integers n.
   c. $A(3, n) = 8 \cdot 2^n - 3$, for all nonnegative integers n.

23. Compute $T(2)$, $T(3)$, $T(4)$, $T(5)$, $T(6)$, and $T(7)$ for the "function" T defined after Example 5.9.8.

24. Student A tries to define a function $F : \mathbf{Z}^+ \to \mathbf{Z}$ by the rule

$$F(n) = \begin{cases} 1 & \text{if } n \text{ is } 1 \\ F\left(\dfrac{n}{2}\right) & \text{if } n \text{ is even} \\ 1 + F(5n - 9) & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

for all integers $n \geq 1$. Student B claims that F is not well defined. Justify student B's claim.

25. Student C tries to define a function $G : \mathbf{Z}^+ \to \mathbf{Z}$ by the rule

$$G(n) = \begin{cases} 1 & \text{if } n \text{ is } 1 \\ G\left(\dfrac{n}{2}\right) & \text{if } n \text{ is even} \\ 2 + G(3n - 5) & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

for all integers $n \geq 1$. Student D claims that G is not well defined. Justify student D's claim.