Exercise Set 6.1*

1. In each of (a)–(f), answer the following questions: Is $A \subseteq B$? Is $B \subseteq A$? Is either A or B a proper subset of the other?

   a. $A = \{2, \{2\}, (\sqrt{2})^2\}$, $B = \{2, \{2\}, \{\{2\}\}\}$

   b. $A = \{3, \sqrt{5^2 - 4^2}, 24 \bmod 7\}$, $B = \{8 \bmod 5\}$

   c. $A = \{\{1, 2\}, \{2, 3\}\}$, $B = \{1, 2, 3\}$

   d. $A = \{a, b, c\}$, $B = \{\{a\}, \{b\}, \{c\}\}$

   e. $A = \{\sqrt{16}, \{4\}\}$, $B = \{4\}$

   f. $A = \{x \in \mathbf{R} \mid \cos x \in \mathbf{Z}\}$, $B = \{x \in \mathbf{R} \mid \sin x \in \mathbf{Z}\}$

2. Complete the proof from Example 6.1.3: Prove that $B \subseteq A$ where

$$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$$

and

$$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$$

* For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ✶ signals that an exercise is more challenging than usual.

3. Let sets R, S, and T be defined as follows:
$$R = \{x \in \mathbf{Z} \mid x \text{ is divisible by } 2\}$$
$$S = \{y \in \mathbf{Z} \mid y \text{ is divisible by } 3\}$$
$$T = \{z \in \mathbf{Z} \mid z \text{ is divisible by } 6\}$$
   a. Is $R \subseteq T$? Explain.
   b. Is $T \subseteq R$? Explain.
   c. Is $T \subseteq S$? Explain.

4. Let $A = \{n \in \mathbf{Z} \mid n = 5r$ for some integer $r\}$ and $B = \{m \in \mathbf{Z} \mid m = 20s$ for some integer $s\}$.
   a. Is $A \subseteq B$? Explain. b. Is $B \subseteq A$? Explain.

5. Let $C = \{n \in \mathbf{Z} \mid n = 6r - 5$ for some integer $r\}$ and $D = \{m \in \mathbf{Z} \mid m = 3s + 1$ for some integer $s\}$.
   Prove or disprove each of the following statements.
   a. $C \subseteq D$ b. $D \subseteq C$

6. Let $A = \{x \in \mathbf{Z} \mid x = 5a + 2$ for some integer $a\}$, $B = \{y \in \mathbf{Z} \mid y = 10b - 3$ for some integer $b\}$, and $C = \{z \in \mathbf{Z} \mid z = 10c + 7$ for some integer $c\}$.
   Prove or disprove each of the following statements.
   a. $A \subseteq B$ b. $B \subseteq A$ *H* **c.** $B = C$

7. Let $A = \{x \in \mathbf{Z} \mid x = 6a + 4$ for some integer $a\}$, $B = \{y \in \mathbf{Z} \mid y = 18b - 2$ for some integer $b\}$, and $C = \{z \in \mathbf{Z} \mid z = 18c + 16$ for some integer $c\}$.
   Prove or disprove each of the following statements.
   a. $A \subseteq B$ b. $B \subseteq A$ c. $B = C$

8. Write in words how to read each of the following out loud. Then write the shorthand notation for each set.
   a. $\{x \in U \mid x \in A \text{ and } x \in B\}$
   b. $\{x \in U \mid x \in A \text{ or } x \in B\}$
   c. $\{x \in U \mid x \in A \text{ and } x \notin B\}$
   d. $\{x \in U \mid x \notin A\}$

9. Complete the following sentences without using the symbols $\cup$, $\cap$, or $-$.
   a. $x \notin A \cup B$ if, and only if, ______.
   b. $x \notin A \cap B$ if, and only if, ______.
   c. $x \notin A - B$ if, and only if, ______.

10. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{3, 6, 9\}$, and $C = \{2, 4, 6, 8\}$. Find each of the following:
    a. $A \cup B$ **b.** $A \cap B$ **c.** $A \cup C$ **d.** $A \cap C$
    e. $A - B$ f. $B - A$ g. $B \cup C$ h. $B \cap C$

11. Let the universal set be the set $\mathbf{R}$ of all real numbers and let $A = \{x \in \mathbf{R} \mid 0 < x \le 2\}$, $B = \{x \in \mathbf{R} \mid 1 \le x < 4\}$, and $C = \{x \in \mathbf{R} \mid 3 \le x < 9\}$. Find each of the following:
    a. $A \cup B$ b. $A \cap B$ c. A^c d. $A \cup C$
    e. $A \cap C$ f. B^c g. $A^c \cap B^c$
    h. $A^c \cup B^c$ i. $(A \cap B)^c$ j. $(A \cup B)^c$

12. Let the universal set be the set $\mathbf{R}$ of all real numbers and let $A = \{x \in \mathbf{R} \mid -3 \le x \le 0\}$, $B = \{x \in \mathbf{R} \mid -1 < x < 2\}$, and $C = \{x \in \mathbf{R} \mid 6 < x \le 8\}$. Find each of the following:
    a. $A \cup B$ b. $A \cap B$ c. A^c d. $A \cup C$
    e. $A \cap C$ f. B^c g. $A^c \cap B^c$
    h. $A^c \cup B^c$ i. $(A \cap B)^c$ j. $(A \cup B)^c$

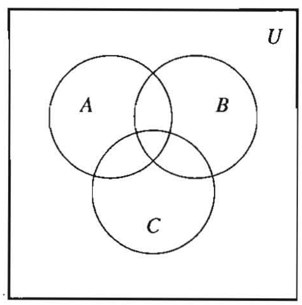
13. Indicate which of the following relationships are true and which are false:
    a. $\mathbf{Z}^+ \subseteq \mathbf{Q}$ **b.** $\mathbf{R}^- \subseteq \mathbf{Q}$
    c. $\mathbf{Q} \subseteq \mathbf{Z}$ **d.** $\mathbf{Z}^- \cup \mathbf{Z}^+ = \mathbf{Z}$
    e. $\mathbf{Z}^- \cap \mathbf{Z}^+ = \emptyset$ f. $\mathbf{Q} \cap \mathbf{R} = \mathbf{Q}$
    g. $\mathbf{Q} \cup \mathbf{Z} = \mathbf{Q}$ h. $\mathbf{Z}^+ \cap \mathbf{R} = \mathbf{Z}^+$
    i. $\mathbf{Z} \cup \mathbf{Q} = \mathbf{Z}$

14. In each of the following, draw a Venn diagram for sets A, B, and C that satisfy the given conditions:
    a. $A \subseteq B$; $C \subseteq B$; $A \cap C = \emptyset$
    b. $C \subseteq A$; $B \cap C = \emptyset$

15. Draw Venn diagrams to describe sets A, B, and C that satisfy the given conditions.
    a. $A \cap B = \emptyset$, $A \subseteq C$, $C \cap B \neq \emptyset$
    b. $A \subseteq B$, $C \subseteq B$, $A \cap C \neq \emptyset$
    c. $A \cap B \neq \emptyset$, $B \cap C \neq \emptyset$, $A \cap C = \emptyset$, $A \not\subseteq B$, $C \not\subseteq B$

16. Let $A = \{a, b, c\}$, $B = \{b, c, d\}$, and $C = \{b, c, e\}$.
    a. Find $A \cup (B \cap C)$, $(A \cup B) \cap C$, and $(A \cup B) \cap (A \cup C)$. Which of these sets are equal?
    b. Find $A \cap (B \cup C)$, $(A \cap B) \cup C$, and $(A \cap B) \cup (A \cap C)$. Which of these sets are equal?
    c. Find $(A - B) - C$ and $A - (B - C)$. Are these sets equal?

17. Consider the Venn diagram shown below. For each of (a)–(f), copy the diagram and shade the region corresponding to the indicated set.
    a. $A \cap B$ b. $B \cup C$ c. A^c
    d. $A - (B \cup C)$ e. $(A \cup B)^c$ f. $A^c \cap B^c$



18. **a.** Is the number 0 in $\emptyset$? Why? **b.** Is $\emptyset = \{\emptyset\}$? Why?
    c. Is $\emptyset \in \{\emptyset\}$? Why? d. Is $\emptyset \in \emptyset$? Why?

19. Let $A_i = \{i, i^2\}$ for all integers $i = 1, 2, 3, 4$.
    a. $A_1 \cup A_2 \cup A_3 \cup A_4 = ?$
    b. $A_1 \cap A_2 \cap A_3 \cap A_4 = ?$
    c. Are A_1, A_2, A_3, and A_4 mutually disjoint? Explain.

20. Let $B_i = \{x \in \mathbf{R} \mid 0 \le x \le i\}$ for all integers $i = 1, 2, 3, 4$.
    a. $B_1 \cup B_2 \cup B_3 \cup B_4 = ?$
    b. $B_1 \cap B_2 \cap B_3 \cap B_4 = ?$
    c. Are B_1, B_2, B_3, and B_4 mutually disjoint? Explain.

21. Let $C_i = \{i, -i\}$ for all nonnegative integers i.
    a. $\bigcup_{i=0}^{4} C_i = ?$ b. $\bigcap_{i=0}^{4} C_i = ?$

c. Are $C_0, C_1, C_2, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=0}^{n} C_i = ?$ e. $\bigcap_{i=0}^{n} C_i = ?$

f. $\bigcup_{i=0}^{\infty} C_i = ?$ g. $\bigcap_{i=0}^{\infty} C_i = ?$

22. Let $D_i = \{x \in \mathbf{R} \mid -i \leq x \leq i\} = [-i, i]$ for all nonnegative integers i.

a. $\bigcup_{i=0}^{4} D_i = ?$ b. $\bigcap_{i=0}^{4} D_i = ?$

c. Are $D_0, D_1, D_2, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=0}^{n} D_i = ?$ e. $\bigcap_{i=0}^{n} D_i = ?$

f. $\bigcup_{i=0}^{\infty} D_i = ?$ g. $\bigcap_{i=0}^{\infty} D_i = ?$

23. Let $V_i = \left\{x \in \mathbf{R} \mid -\frac{1}{i} \leq x \leq \frac{1}{i}\right\} = \left[-\frac{1}{i}, \frac{1}{i}\right]$ for all positive integers i.

a. $\bigcup_{i=1}^{4} V_i = ?$ b. $\bigcap_{i=1}^{4} V_i = ?$

c. Are $V_1, V_2, V_3, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=1}^{n} V_i = ?$ e. $\bigcap_{i=1}^{n} V_i = ?$

f. $\bigcup_{i=1}^{\infty} V_i = ?$ g. $\bigcap_{i=1}^{\infty} V_i = ?$

24. Let $W_i = \{x \in \mathbf{R} \mid x > i\} = (i, \infty)$ for all nonnegative integers i.

a. $\bigcup_{i=0}^{4} W_i = ?$ b. $\bigcap_{i=0}^{4} W_i = ?$

c. Are $W_0, W_1, W_2, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=0}^{n} W_i = ?$ e. $\bigcap_{i=0}^{n} W_i = ?$

f. $\bigcup_{i=0}^{\infty} W_i = ?$ g. $\bigcap_{i=0}^{\infty} W_i = ?$

25. Let $R_i = \left\{x \in \mathbf{R} \mid 1 \leq x \leq 1 + \frac{1}{i}\right\} = \left[1, 1 + \frac{1}{i}\right]$ for all positive integers i.

a. $\bigcup_{i=1}^{4} R_i = ?$ b. $\bigcap_{i=1}^{4} R_i = ?$

c. Are $R_1, R_2, R_3, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=1}^{n} R_i = ?$ e. $\bigcap_{i=1}^{n} R_i = ?$

f. $\bigcup_{i=1}^{\infty} R_i = ?$ g. $\bigcap_{i=1}^{\infty} R_i = ?$

26. Let $S_i = \left\{x \in \mathbf{R} \mid 1 < x < 1 + \frac{1}{i}\right\} = \left(1, 1 + \frac{1}{i}\right)$ for all positive integers i.

a. $\bigcup_{i=1}^{4} S_i = ?$ b. $\bigcap_{i=1}^{4} S_i = ?$

c. Are $S_1, S_2, S_3, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=1}^{n} S_i = ?$ e. $\bigcap_{i=1}^{n} S_i = ?$

f. $\bigcup_{i=1}^{\infty} S_i = ?$ g. $\bigcap_{i=1}^{\infty} S_i = ?$

27. **a.** Is $\{\{a, d, e\}, \{b, c\}, \{d, f\}\}$ a partition of $\{a, b, c, d, e, f\}$?

b. Is $\{\{w, x, v\}, \{u, y, q\}, \{p, z\}\}$ a partition of $\{p, q, u, v, w, x, y, z\}$?

c. Is $\{\{5, 4\}, \{7, 2\}, \{1, 3, 4\}, \{6, 8\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8\}$?

d. Is $\{\{3, 7, 8\}, \{2, 9\}, \{1, 4, 5\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

e. Is $\{\{1, 5\}, \{4, 7\}, \{2, 8, 6, 3\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8\}$?

28. Let E be the set of all even integers and O the set of all odd integers. Is $\{E, O\}$ a partition of $\mathbf{Z}$, the set of all integers? Explain your answer.

29. Let $\mathbf{R}$ be the set of all real numbers. Is $\{\mathbf{R}^+, \mathbf{R}^-, \{0\}\}$ a partition of $\mathbf{R}$? Explain your answer.

30. Let $\mathbf{Z}$ be the set of all integers and let

$$A_0 = \{n \in \mathbf{Z} \mid n = 4k, \text{ for some integer } k\},$$
$$A_1 = \{n \in \mathbf{Z} \mid n = 4k + 1, \text{ for some integer } k\},$$
$$A_2 = \{n \in \mathbf{Z} \mid n = 4k + 2, \text{ for some integer } k\}, \text{ and}$$
$$A_3 = \{n \in \mathbf{Z} \mid n = 4k + 3, \text{ for some integer } k\}.$$

Is $\{A_0, A_1, A_2, A_3\}$ a partition of $\mathbf{Z}$? Explain your answer.

31. Suppose $A = \{1, 2\}$ and $B = \{2, 3\}$. Find each of the following:

a. $\mathscr{P}(A \cap B)$ b. $\mathscr{P}(A)$

c. $\mathscr{P}(A \cup B)$ d. $\mathscr{P}(A \times B)$

32. **a.** Suppose $A = \{1\}$ and $B = \{u, v\}$. Find $\mathscr{P}(A \times B)$.

b. Suppose $X = \{a, b\}$ and $Y = \{x, y\}$. Find $\mathscr{P}(X \times Y)$.

33. a. Find $\mathscr{P}(\emptyset)$. **b.** Find $\mathscr{P}(\mathscr{P}(\emptyset))$.

c. Find $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset)))$.

34. Let $A_1 = \{1, 2, 3\}$, $A_2 = \{u, v\}$, and $A_3 = \{m, n\}$. Find each of the following sets:

a. $A_1 \times (A_2 \times A_3)$ b. $(A_1 \times A_2) \times A_3$

c. $A_1 \times A_2 \times A_3$

35. Let $A = \{a, b\}$, $B = \{1, 2\}$, and $C = \{2, 3\}$. Find each of the following sets.

a. $A \times (B \cup C)$ **b.** $(A \times B) \cup (A \times C)$

c. $A \times (B \cap C)$ d. $(A \times B) \cap (A \times C)$

36. Trace the action of Algorithm 6.1.1 on the variables i, j, *found*, and *answer* for $m = 3$, $n = 3$, and sets A and B represented as the arrays $a[1] = u, a[2] = v, a[3] = w$, $b[1] = w, b[2] = u$, and $b[3] = v$.

37. Trace the action of Algorithm 6.1.1 on the variables i, j, *found*, and *answer* for $m = 4$, $n = 4$, and sets A and B represented as the arrays $a[1] = u, a[2] = v, a[3] = w$, $a[4] = x, b[1] = r, b[2] = u, b[3] = y, b[4] = z$.

38. Write an algorithm to determine whether a given element x belongs to a given set, which is represented as an array $a[1], a[2], \ldots, a[n]$.