Exercise Set 6.2

1. **a.** To say that an element is in $A \cap (B \cup C)$ means that it is in $\underline{\quad(1)\quad}$ and in $\underline{\quad(2)\quad}$.

   b. To say that an element is in $(A \cap B) \cup C$ means that it is in $\underline{\quad(1)\quad}$ or in $\underline{\quad(2)\quad}$.

   c. To say that an element is in $A - (B \cap C)$ means that it is in $\underline{\quad(1)\quad}$ and not in $\underline{\quad(2)\quad}$.

2. The following are two proofs that for all sets A and B, $A - B \subseteq A$. The first is less formal, and the second is more formal. Fill in the blanks.

   a. Proof: Suppose A and B are any sets. To show that $A - B \subseteq A$, we must show that every element in $\underline{\quad(1)\quad}$ is in $\underline{\quad(2)\quad}$. But any element in $A - B$ is in $\underline{\quad(3)\quad}$ and not in $\underline{(4)}$ (by definition of $A - B$). In particular, such an element is in A.

b. **Proof:** Suppose A and B are any sets and $x \in A - B$. *[We must show that* $\underline{(1)}$*.]* By definition of set difference, $x \in \underline{(2)}$ and $x \notin \underline{(3)}$. In particular, $x \in \underline{(4)}$ *[which is what was to be shown].*

3. The following is a proof that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Fill in the blanks.

   Proof: Suppose A, B, and C are sets and $A \subseteq B$ and $B \subseteq C$. To show that $A \subseteq C$, we must show that every element in $\underline{(a)}$ is in $\underline{(b)}$. But given any element in A, that element is in $\underline{(c)}$ (because $A \subseteq B$), and so that element is also in $\underline{(d)}$ (because $\underline{(e)}$). Hence $A \subseteq C$.

4. The following is a proof that for all sets A and B, if $A \subseteq B$, then $A \cup B \subseteq B$. Fill in the blanks.

   Proof: Suppose A and B are any sets and $A \subseteq B$. *[We must show that* $\underline{(a)}$*.]* Let $x \in \underline{(b)}$. *[We must show that* $\underline{(c)}$*.]* By definition of union, $x \in \underline{(d)}$ $\underline{(e)}$ $x \in \underline{(f)}$. In case $x \in \underline{(g)}$, then since $A \subseteq B$, $x \in \underline{(h)}$. In case $x \in B$, then clearly $x \in B$. So in either case, $x \in \underline{(i)}$ *[as was to be shown].*

5. Prove that for all sets A and B, $(B - A) = B \cap A^c$.

H 6. The following is a proof that for any sets A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Fill in the blanks.

   Proof: Suppose A, B, and C are any sets.

   (1) *Proof that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:*
   Let $x \in A \cap (B \cup C)$. *[We must show that $x \in$* $\underline{(a)}$*.]* By definition of intersection, $x \in \underline{(b)}$ and $x \in \underline{(c)}$. Thus $x \in A$ and, by definition of union, $x \in B$ or $\underline{(d)}$.

   Case 1 ($x \in A$ and $x \in B$): In this case, by definition of intersection, $x \in \underline{(e)}$, and so, by definition of union, $x \in (A \cap B) \cup (A \cap C)$.

   Case 2 ($x \in A$ and $x \in C$): In this case, $\underline{(f)}$.
   Hence in either case, $x \in (A \cap B) \cup (A \cap C)$ *[as was to be shown].*
   [So $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ by definition of subset.]

   (2) $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$:
   Let $x \in (A \cap B) \cup (A \cap C)$. *[We must show that* $\underline{(a)}$*.]* By definition of union, $x \in A \cap B$ $\underline{(b)}$ $x \in A \cap C$.

   Case 1 ($x \in A \cap B$): In this case, by definition of intersection, $x \in A$ $\underline{(c)}$ $x \in B$. Since $x \in B$, then by definition of union, $x \in B \cup C$. Hence $x \in A$ and $x \in B \cup C$, and so, by definition of intersection, $x \in \underline{(d)}$.

   Case 2 ($x \in A \cap C$): In this case, $\underline{(e)}$.
   In either case, $x \in A \cap (B \cup C)$ *[as was to be shown]. [Thus $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ by definition of subset.]*

   (3) *Conclusion:* *[Since both subset relations have been proved, it follows, by definition of set equality, that* $\underline{(a)}$*.]*

Use an element argument to prove each statement in 7–19. Assume that all sets are subsets of a universal set U.

H 7. For all sets A and B, $(A \cap B)^c = A^c \cup B^c$.

8. For all sets A and B, $(A \cap B) \cup (A \cap B^c) = A$.

H 9. For all sets A, B, and C,

$$(A - B) \cup (C - B) = (A \cup C) - B.$$

10. For all sets A, B, and C,

$$(A - B) \cap (C - B) = (A \cap C) - B.$$

H 11. For all sets A and B, $A \cup (A \cap B) = A$.

12. For all sets A, $A \cup \emptyset = A$.

13. For all sets A, B, and C, if $A \subseteq B$ then $A \cap C \subseteq B \cap C$.

14. For all sets A, B, and C, if $A \subseteq B$ then $A \cup C \subseteq B \cup C$.

15. For all sets A and B, if $A \subseteq B$ then $B^c \subseteq A^c$.

H 16. For all sets A, B, and C, if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

17. For all sets A, B, and C, if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.

18. For all sets A, B, and C,

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

19. For all sets A, B, and C,

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

20. Find the mistake in the following "proof" that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

   "**Proof:** Suppose A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Since $A \subseteq B$, there is an element x such that $x \in A$ and $x \in B$. Since $B \subseteq C$, there is an element x such that $x \in B$ and $x \in C$. Hence there is an element x such that $x \in A$ and $x \in C$ and so $A \subseteq C$."

H 21. Find the mistake in the following "proof."

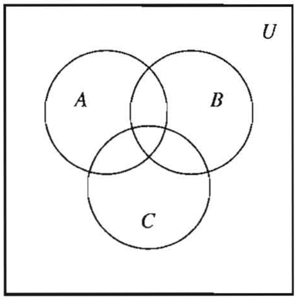
   "**Theorem:**" For all sets A and B, $A^c \cup B^c \subseteq (A \cup B)^c$.

   "**Proof:** Suppose A and B are sets, and $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$ by definition of union. It follows that $x \notin A$ or $x \notin B$ by definition of complement, and so $x \notin A \cup B$ by definition of union. Thus $x \in (A \cup B)^c$ by definition of complement, and hence $A^c \cup B^c \subseteq (A \cup B)^c$."

22. Find the mistake in the following "proof" that for all sets A and B, $(A - B) \cup (A \cap B) \subseteq A$.

   "**Proof:** Suppose A and B are sets, and suppose $x \in (A - B) \cup (A \cap B)$. If $x \in A$ then $x \in A - B$. Then, by definition of difference, $x \in A$ and $x \notin B$. Hence $x \in A$, and so $(A - B) \cup (A \cap B) \subseteq A$ by definition of subset."

23. Consider the Venn diagram below.



a. Illustrate one of the distributive laws by shading in the region corresponding to $A \cup (B \cap C)$ on one copy of the diagram and $(A \cup B) \cap (A \cup C)$ on another.
b. Illustrate the other distributive law by shading in the region corresponding to $A \cap (B \cup C)$ on one copy of the diagram and $(A \cap B) \cup (A \cap C)$ on another.
c. Illustrate one of De Morgan's laws by shading in the region corresponding to $(A \cup B)^c$ on one copy of the diagram and $A^c \cap B^c$ on the other. (Leave the set C out of your diagrams.)
d. Illustrate the other De Morgan's law by shading in the region corresponding to $(A \cap B)^c$ on one copy of the diagram and $A^c \cup B^c$ on the other. (Leave the set C out of your diagrams.)

24. Fill in the blanks in the following proof that for all sets A and B, $(A - B) \cap (B - A) = \emptyset$.

Proof: Let A and B be any sets and supppose $(A - B) \cap (B - A) \neq \emptyset$. That is, suppose there were an element x in $\underline{(a)}$. By definition of $\underline{(b)}$, $x \in A - B$ and $x \in \underline{(c)}$. Then by definition of set difference, $x \in A$ and $x \notin B$ and $x \in \underline{(d)}$ and $x \notin \underline{(e)}$. In particular $x \in A$ and $x \notin \underline{(f)}$, which is a contradiction. Hence *[the supposition that $(A - B) \cap (B - A) \neq \emptyset$ is false, and so]* $\underline{(g)}$.

Use the element method for proving a set equals the empty set to prove each statement in 25–35. Assume that all sets are subsets of a universal set U.

25. For all sets A and B, $(A \cap B) \cap (A \cap B^c) = \emptyset$.

26. For all sets A, B, and C,

$$(A - C) \cap (B - C) \cap (A - B) = \emptyset.$$

27. For all subsets A of a universal set U, $A \cap A^c = \emptyset$.

28. If U denotes a universal set, then $U^c = \emptyset$.

29. For all sets A, $A \times \emptyset = \emptyset$.

30. For all sets A and B, if $A \subseteq B$ then $A \cap B^c = \emptyset$.

31. For all sets A and B, if $B \subseteq A^c$ then $A \cap B = \emptyset$.

32. For all sets A, B, and C, if $A \subseteq B$ and $B \cap C = \emptyset$ then $A \cap C = \emptyset$.

33. For all sets A, B, and C, if $C \subseteq B - A$, then $A \cap C = \emptyset$.

34. For all sets A, B, and C,

$$\text{if } B \cap C \subseteq A, \text{ then } (C - A) \cap (B - A) = \emptyset.$$

35. For all sets A, B, C, and D,

$$\text{if } A \cap C = \emptyset \text{ then } (A \times B) \cap (C \times D) = \emptyset.$$

Prove each statement in 36–41.

***H* 36.** For all sets A and B,
a. $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$
b. The sets $(A - B)$, $(B - A)$, and $(A \cap B)$ are mutually disjoint.

37. For all integers $n \geq 1$, if A and $B_1, B_2, B_3, \ldots$ are any sets, then

$$A \cap \left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{i=1}^{n} (A \cap B_i).$$

***H* 38.** For all integers $n \geq 1$, if $A_1, A_2, A_3, \ldots$ and B are any sets, then

$$\bigcup_{i=1}^{n} (A_i - B) = \left(\bigcup_{i=1}^{n} A_i\right) - B.$$

39. For all integers $n \geq 1$, if $A_1, A_2, A_3, \ldots$ and B are any sets, then

$$\bigcap_{i=1}^{n} (A_i - B) = \left(\bigcap_{i=1}^{n} A_i\right) - B.$$

40. For all integers $n \geq 1$, if A and $B_1, B_2, B_3, \ldots$ are any sets, then

$$\bigcup_{i=1}^{n} (A \times B_i) = A \times \left(\bigcup_{i=1}^{n} B_i\right).$$

41. For all integers $n \geq 1$, if A and $B_1, B_2, B_3, \ldots$ are any sets, then

$$\bigcap_{i=1}^{n} (A \times B_i) = A \times \left(\bigcap_{i=1}^{n} B_i\right).$$