Exercise Set 6.3

For each of 1–4 find a counterexample to show that the statement is false. Assume all sets are subsets of a universal set U.

1. For all sets A, B, and C, $(A \cap B) \cup C = A \cap (B \cup C)$.
2. For all sets A and B, $(A \cup B)^c = A^c \cup B^c$.
3. For all sets A, B, and C, if $A \not\subseteq B$ and $B \not\subseteq C$ then $A \not\subseteq C$.
4. For all sets A, B, and C, if $B \cap C \subseteq A$ then $(A - B) \cap (A - C) = \emptyset$.

For each of 5–21 prove each statement that is true and find a counterexample for each statement that is false. Assume all sets are subsets of a universal set U.

5. For all sets A, B, and C, $A - (B - C) = (A - B) - C$.
6. For all sets A and B, $A \cap (A \cup B) = A$.
7. For all sets A, B, and C,
$$(A - B) \cap (C - B) = A - (B \cup C).$$
8. For all sets A and B, if $A^c \subseteq B$ then $A \cup B = U$.
9. For all sets A, B, and C, if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.
10. For all sets A and B, if $A \subseteq B$ then $A \cap B^c = \emptyset$.

H 11. For all sets A, B, and C, if $A \subseteq B$ then $A \cap (B \cap C)^c = \emptyset$.

H 12. For all sets A, B, and C,
$$A \cap (B - C) = (A \cap B) - (A \cap C).$$

13. For all sets A, B, and C,
$$A \cup (B - C) = (A \cup B) - (A \cup C).$$

H 14. For all sets A, B, and C, if $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$, then $A \subseteq B$.

H 15. For all sets A, B, and C, if $A \cap C = B \cap C$ and $A \cup C = B \cup C$, then $A = B$.

16. For all sets A and B, if $A \cap B = \emptyset$ then $A \times B = \emptyset$.
17. For all sets A and B, if $A \subseteq B$ then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.
18. For all sets A and B, $\mathscr{P}(A \cup B) \subseteq \mathscr{P}(A) \cup \mathscr{P}(B)$.

H 19. For all sets A and B, $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$.

20. For all sets A and B, $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.
21. For all sets A and B, $\mathscr{P}(A \times B) = \mathscr{P}(A) \times \mathscr{P}(B)$.
22. Write a negation for each of the following statements. Indicate which is true, the statement or its negation. Justify your answers.
    a. $\forall$ sets S, $\exists$ a set T such that $S \cap T = \emptyset$.
    b. $\exists$ a set S such that $\forall$ sets T, $S \cup T = \emptyset$.

H 23. Let $S = \{a, b, c\}$ and for each integer $i = 0, 1, 2, 3$, let S_i be the set of all subsets of S that have i elements. List the elements in S_0, S_1, S_2, and S_3. Is $\{S_0, S_1, S_2, S_3\}$ a partition of $\mathscr{P}(S)$?

24. Let $S = \{a, b, c\}$ and let S_a be the set of all subsets of S that contain a, let S_b be the set of all subsets of S that contain b, let S_c be the set of all subsets of S that contain c, and let $S_\emptyset$ be the set whose only element is $\emptyset$. Is $\{S_a, S_b, S_c, S_\emptyset\}$ a partition of $\mathscr{P}(S)$?

25. Let $A = \{t, u, v, w\}$ and let S_1 be the set of all subsets of A that do not contain w and S_2 the set of all subsets of A that contain w.
    a. Find S_1. **b.** Find S_2. **c.** Are S_1 and S_2 disjoint?
    d. Compare the sizes of S_1 and S_2.
    e. How many elements are in $S_1 \cup S_2$?
    f. What is the relation between $S_1 \cup S_2$ and $\mathscr{P}(A)$?

***H*★26.** The following problem, devised by Ginger Bolton, appeared in the January 1989 issue of the *College Mathematics Journal* (Vol. 20, No. 1, p. 68): Given a positive integer $n \geq 2$, let S be the set of all nonempty subsets of $\{2, 3, \ldots, n\}$. For each $S_i \in S$, let P_i be the product of the elements of S_i. Prove or disprove that

$$\sum_{i=1}^{2^{n-1}-1} P_i = \frac{(n+1)!}{2} - 1.$$

In 27 and 28 supply a reason for each step in the derivation.

27. For all sets A, B, and C,

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$$

Proof: Suppose A, B, and C are any sets. Then

$$\begin{aligned}(A \cup B) \cap C &= C \cap (A \cup B) && \text{by } \underline{\ (a)\ } \\ &= (C \cap A) \cup (C \cap B) && \text{by } \underline{\ (b)\ } \\ &= (A \cap C) \cup (B \cap C) && \text{by } \underline{\ (c)\ }.\end{aligned}$$

***H* 28.** For all sets A, B, and C, $(A \cup B) - (C - A) = A \cup (B - C)$.

Proof: Suppose A, B, and C are any sets. Then

$$\begin{aligned}(A \cup B) - (C - A) &= (A \cup B) \cap (C - A)^c && \text{by } \underline{\ (a)\ } \\ &= (A \cup B) \cap (C \cap A^c)^c && \text{by } \underline{\ (b)\ } \\ &= (A \cup B) \cap (A^c \cap C)^c && \text{by } \underline{\ (c)\ } \\ &= (A \cup B) \cap ((A^c)^c \cup C^c) && \text{by } \underline{\ (d)\ } \\ &= (A \cup B) \cap (A \cup C^c) && \text{by } \underline{\ (e)\ } \\ &= A \cup (B \cap C^c) && \text{by } \underline{\ (f)\ } \\ &= A \cup (B - C) && \text{by } \underline{\ (g)\ }.\end{aligned}$$

***H* 29.** Some steps are missing from the following proof that for all sets $(A \cup B) - C = (A - C) \cup (B - C)$. Indicate what they are, and then write the proof correctly.

Proof: Let A, B, and C be any sets. Then

$$\begin{aligned}(A \cup B) - C &= (A \cup B) \cap C^c && \text{by the set difference law} \\ &= (A \cap C^c) \cup (B \cap C^c) && \text{by the distributive law} \\ &= (A - C) \cup (B - C) && \text{by the set difference law}\end{aligned}$$

In 30–40, construct an algebraic proof for the given statement. Cite a property from Theorem 6.2.2 for every step.

30. For all sets A, B, and C,

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

31. For all sets A and B, $A \cup (B - A) = A \cup B$.

32. For all sets A and B, $(A - B) \cup (A \cap B) = A$.

33. For all sets A and B, $(A - B) \cap (A \cap B) = \emptyset$.

34. For all sets A, B, and C,

$$(A - B) - C = A - (B \cup C).$$

35. For all sets A and B, $A - (A - B) = A \cap B$.

36. For all sets A and B, $((A^c \cup B^c) - A)^c = A$.

37. For all sets A and B, $(B^c \cup (B^c - A))^c = B$.

38. For all sets A and B, $A - (A \cap B) = A - B$.

***H* 39.** For all sets A and B,

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

40. For all sets A, B, and C,

$$(A - B) - (B - C) = A - B.$$

In 41–43 simplify the given expression. Cite a property from Theorem 6.2.2 for every step.

***H* 41.** $A \cap ((B \cup A^c) \cap B^c)$

42. $(A - (A \cap B)) \cap (B - (A \cap B))$

43. $((A \cap (B \cup C)) \cap (A - B)) \cap (B \cup C^c)$

44. Consider the following set property: For all sets A and B, $A - B$ and B are disjoint.
    a. Use an element argument to derive the property.
    b. Use an algebraic argument to derive the property (by applying properties from Theorem 6.2.2).
    c. Comment on which method you found easier.

45. Consider the following set property: For all sets A, B, and C, $(A - B) \cup (B - C) = (A \cup B) - (B \cap C)$.
    a. Use an element argument to derive the property.
    b. Use an algebraic argument to derive the property (by applying properties from Theorem 6.2.2).
    c. Comment on which method you found easier.

Definition: Given sets A and B, the **symmetric difference of A and B,** denoted $A \triangle B$, is

$$A \triangle B = (A - B) \cup (B - A).$$

46. Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{5, 6, 7, 8\}$. Find each of the following sets:
    a. $A \triangle B$ b. $B \triangle C$
    c. $A \triangle C$ d. $(A \triangle B) \triangle C$

Refer to the definition of symmetric difference given above. Prove each of 47–52, assuming that A, B, and C are all subsets of a universal set U.

47. $A \triangle B = B \triangle A$ **48.** $A \triangle \emptyset = A$

49. $A \triangle A^c = U$ 50. $A \triangle A = \emptyset$

***H* 51.** If $A \triangle C = B \triangle C$, then $A = B$.

H **52.** $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

H **53.** Derive the set identity $A \cup (A \cap B) = A$ from the properties listed in Theorem 6.2.2(1)–(9). Start by showing that for all subsets B of a universal set U, $U \cup B = U$. Then intersect both sides with A and deduce the identity.

54. Derive the set identity $A \cap (A \cup B) = A$ from the properties listed in Theorem 6.2.2(1)–(9). Start by showing that for all subsets B of a universal set U, $\emptyset = \emptyset \cap B$. Then take the union of both sides with A and deduce the identity.

Answers for Test Yourself

1. make the left-hand side unequal to the right-hand side (*Or: result in different values on the two sides of the equation*) 2. *cite one of the properties from Theorem 6.2.2 (Or: give a reason)* 3. *exactly*

6.4 Boolean Algebras, Russell's Paradox, and the Halting Problem

From the paradise created for us by Cantor, no one will drive us out.
— David Hilbert (1862–1943)

Table 6.4.1 summarizes the main features of the logical equivalences from Theorem 2.1.1 and the set properties from Theorem 6.2.2. Notice how similar the entries in the two columns are.

Logical Equivalences	**Set Properties**
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) \equiv A \cup (B \cup C)$ b. $A \cap (B \cap C) \equiv A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$
a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge \sim p \equiv \mathbf{c}$	a. $A \cup A^c = U$ b. $A \cap A^c = \emptyset$
$\sim(\sim p) \equiv p$	$(A^c)^c = A$
a. $p \vee p \equiv p$ b. $p \wedge p \equiv p$	a. $A \cup A = A$ b. $A \cap A = A$
a. $p \vee \mathbf{t} \equiv \mathbf{t}$ b. $p \wedge \mathbf{c} \equiv \mathbf{c}$	a. $A \cup U = U$ b. $A \cap \emptyset = \emptyset$
a. $\sim(p \vee q) \equiv \sim p \wedge \sim q$ b. $\sim(p \wedge q) \equiv \sim p \vee \sim q$	a. $(A \cup B)^c = A^c \cap B^c$ b. $(A \cap B)^c = A^c \cup B^c$
a. $p \vee (p \wedge q) \equiv p$ b. $p \wedge (p \vee q) \equiv p$	a. $A \cup (A \cap B) \equiv A$ b. $A \cap (A \cup B) \equiv A$
a. $\sim \mathbf{t} \equiv \mathbf{c}$ b. $\sim \mathbf{c} \equiv \mathbf{t}$	a. $U^c = \emptyset$ b. $\emptyset^c = U$

Table 6.4.1