Exercise Set 6.4

In 1–3 assume that B is a Boolean algebra with operations $+$ and $\cdot$. Give the reasons needed to fill in the blanks in the proofs, but do not use any parts of Theorem 6.4.1 unless they have already been proved. You may use any part of the definition of a Boolean algebra and the results of previous exercises, however.

1. For all a in B, $a \cdot a = a$.

Proof: Let a be any element of B. Then

$$\begin{aligned} a &= a \cdot 1 && \underline{\text{(a)}} \\ &= a \cdot (a + \overline{a}) && \underline{\text{(b)}} \\ &= (a \cdot a) + (a \cdot \overline{a}) && \underline{\text{(c)}} \\ &= (a \cdot a) + 0 && \underline{\text{(d)}} \\ &= a \cdot a && \underline{\text{(e)}}. \end{aligned}$$

2. For all a in B, $a + 1 = 1$.

Proof: Let a be any element of B. Then

$$\begin{aligned} a + 1 &= a + (a + \overline{a}) && \underline{\text{(a)}} \\ &= (a + a) + \overline{a} && \underline{\text{(b)}} \\ &= a + \overline{a} && \text{by Example 6.4.2} \\ &= 1 && \underline{\text{(c)}}. \end{aligned}$$

3. For all a and b in B, $(a + b) \cdot a = a$.

Proof: Let a and b be any elements of B. Then

$$\begin{aligned} (a + b) \cdot a &= a \cdot (a + b) && \underline{\text{(a)}} \\ &= a \cdot a + a \cdot b && \underline{\text{(b)}} \\ &= a + a \cdot b && \underline{\text{(c)}} \\ &= a \cdot 1 + a \cdot b && \underline{\text{(d)}} \\ &= a \cdot (1 + b) && \underline{\text{(e)}} \\ &= a \cdot (b + 1) && \underline{\text{(f)}} \\ &= a \cdot 1 && \text{by exercise 2} \\ &= a && \underline{\text{(g)}}. \end{aligned}$$

In 4–10 assume that B is a Boolean algebra with operations $+$ and $\cdot$. Prove each statement without using any parts of Theorem 6.4.1 unless they have already been proved. You may use any part of the definition of a Boolean algebra and the results of previous exercises, however.

4. For all a in B, $a \cdot 0 = 0$.

5. For all a and b in B, $(a \cdot b) + a = a$.

6. **a.** $\overline{0} = 1$.
   b. $\overline{1} = 0$

7. **a.** There is only one element of B that is an identity for $+$.
H **b.** There is only one element of B that is an identity for $\cdot$.

8. For all a and b in B, $\overline{a \cdot b} = \overline{a} + \overline{b}$. (*Hint:* Prove that $(a \cdot b) + (\overline{a} + \overline{b}) = 1$ and that $(a \cdot b) \cdot (\overline{a} + \overline{b}) = 0$, and use the fact that $a \cdot b$ has a unique complement.)

9. For all a and b in B, $\overline{a + b} = \overline{a} \cdot \overline{b}$.

H **10.** For all x, y, and z in B, if $x + y = x + z$ and $x \cdot y = x \cdot z$, then $y = z$.

11. Let $S = \{0, 1\}$, and define operations $+$ and $\cdot$ on S by the following tables:

+	**0**	**1**
0	0	1
1	1	1

·	**0**	**1**
0	0	0
1	0	1

a. Show that the elements of S satisfy the following properties:
   (i) the commutative law for $+$
   (ii) the commutative law for $\cdot$
   (iii) the associative law for $+$
   (iv) the associative law for $\cdot$
   H (v) the distributive law for $+$ over $\cdot$
   (vi) the distributive law for $\cdot$ over $+$

H **b.** Show that 0 is an identity element for $+$ and that 1 is an identity element for $\cdot$.

c. Define $\overline{0} = 1$ and $\overline{1} = 0$. Show that for all a in S, $a + \overline{a} = 1$ and $a \cdot \overline{a} = 0$. It follows from parts (a)–(c) that S is a Boolean algebra with the operations $+$ and $\cdot$.

H★ **12.** Prove that the associative laws for a Boolean algebra can be omitted from the definition. That is, prove that the associative laws can be derived from the other laws in the definition.

In 13–18 determine whether each sentence is a statement. Explain your answers.

13. This sentence is false.

14. If $1 + 1 = 3$, then $1 = 0$.

15. | The sentence in this box is a lie. |

16. All positive integers with negative squares are prime.

17. This sentence is false or $1 + 1 = 3$.

18. This sentence is false and $1 + 1 = 2$.

19. a. Assuming that the following sentence is a statement, prove that $1 + 1 = 3$:

    If this sentence is true, then $1 + 1 = 3$.

    b. What can you deduce from part (a) about the status of "This sentence is true"? Why? (This example is known as **Löb's paradox.**)

H **20.** The following two sentences were devised by the logician Saul Kripke. While not intrinsically paradoxical, they could be paradoxical under certain circumstances. Describe such circumstances.

(i) Most of Nixon's assertions about Watergate are false.

(ii) Everything Jones says about Watergate is true.

(*Hint:* Suppose Nixon says (ii) and the only utterance Jones makes about Watergate is (i).)

21. Can there exist a computer program that has as output a list of all the computer programs that do not list themselves in their output? Explain your answer.

22. Can there exist a book that refers to all those books and only those books that do not refer to themselves? Explain your answer.

23. Some English adjectives are descriptive of themselves (for instance, the word *polysyllabic* is polysyllabic) whereas others are not (for instance, the word *monosyllabic* is not monosyllabic). The word *heterological* refers to an adjective that does not describe itself. Is *heterological* heterological? Explain your answer.

24. As strange as it may seem, it is possible to give a precise-looking verbal definition of an integer that, in fact, is not a definition at all. The following was devised by an English librarian, G. G. Berry, and reported by Bertrand Russell. Explain how it leads to a contradiction. Let n be "the smallest integer not describable in fewer than 12 English words." (Note that the total number of strings consisting of 11 or fewer English words is finite.)

H **25.** Is there an algorithm which, for a fixed quantity a and any input algorithm X and data set D, can determine whether X prints a when run with data set D? Explain. (This problem is called the **printing problem.**)

26. Use a technique similar to that used to derive Russell's paradox to prove that for any set A, $\mathscr{P}(A) \not\subseteq A$.