Exercise Set 9.4

1. a. If 4 cards are selected from a standard 52-card deck, must at least 2 be of the same suit? Why?

   b. If 5 cards are selected from a standard 52-card deck, must at least 2 be of the same suit? Why?

2. a. If 13 cards are selected from a standard 52-card deck, must at least 2 be of the same denomination? Why?

   b. If 20 cards are selected from a standard 52-card deck, must at least 2 be of the same denomination? Why?

3. A small town has only 500 residents. Must there be 2 residents who have the same birthday? Why?

4. In a group of 700 people, must there be 2 who have the same first and last initials? Why?

5. a. Given any set of four integers, must there be two that have the same remainder when divided by 3? Why?
b. Given any set of three integers, must there be two that have the same remainder when divided by 3? Why?

6. a. Given any set of seven integers, must there be two that have the same remainder when divided by 6? Why?
b. Given any set of seven integers, must there be two that have the same remainder when divided by 8? Why?

H **7.** Let $S = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Suppose six integers are chosen from S. Must there be two integers whose sum is 15? Why?

8. Let $T = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Suppose five integers are chosen from T. Must there be two integers whose sum is 10? Why?

9. a. If seven integers are chosen from between 1 and 12 inclusive, must at least one of them be odd? Why?
b. If ten integers are chosen from between 1 and 20 inclusive, must at least one of them be even? Why?

10. If $n + 1$ integers are chosen from the set

$$\{1, 2, 3, \ldots, 2n\},$$

where n is a positive integer, must at least one of them be odd? Why?

11. If $n + 1$ integers are chosen from the set

$$\{1, 2, 3, \ldots, 2n\},$$

where n is a positive integer, must at least one of them be even? Why?

12. How many cards must you pick from a standard 52-card deck to be sure of getting at least 1 red card? Why?

13. Suppose six pairs of similar-looking boots are thrown together in a pile. How many individual boots must you pick to be sure of getting a matched pair? Why?

14. How many integers from 0 through 60 must you pick in order to be sure of getting at least one that is odd? at least one that is even?

15. If n is a positive integer, how many integers from 0 through $2n$ must you pick in order to be sure of getting at least one that is odd? at least one that is even?

16. How many integers from 1 through 100 must you pick in order to be sure of getting one that is divisible by 5?

17. How many integers must you pick in order to be sure that at least two of them have the same remainder when divided by 7?

18. How many integers must you pick in order to be sure that at least two of them have the same remainder when divided by 15?

19. How many integers from 100 through 999 must you pick in order to be sure that at least two of them have a digit in common? (For example, 256 and 530 have the common digit 5.)

20. a. If repeated divisions by 20,483 are performed, how many distinct remainders can be obtained?
b. When 5/20483 is written as a decimal, what is the maximum length of the repeating section of the representation?

21. When 683/1493 is written as a decimal, what is the maximum length of the repeating section of the representation?

22. Is 0.1010010001000010000001... (where each string of 0's is one longer than the previous one) rational or irrational?

23. Is 56.556655566655556666... (where the strings of 5's and 6's become longer in each repetition) rational or irrational?

24. Show that within any set of thirteen integers chosen from 2 through 40, there are at least two integers with a common divisor greater than 1.

25. In a group of 30 people, must at least 3 have been born in the same month? Why?

26. In a group of 30 people, must at least 4 have been born in the same month? Why?

27. In a group of 2,000 people, must at least 5 have the same birthday? Why?

28. A programmer writes 500 lines of computer code in 17 days. Must there have been at least 1 day when the programmer wrote 30 or more lines of code? Why?

29. A certain college class has 40 students. All the students in the class are known to be from 17 through 34 years of age. You want to make a bet that the class contains at least x students of the same age. How large can you make x and yet be sure to win your bet?

30. A penny collection contains twelve 1967 pennies, seven 1968 pennies, and eleven 1971 pennies. If you are to pick some pennies without looking at the dates, how many must you pick to be sure of getting at least five pennies from the same year?

H **31.** A group of 15 executives are to share 5 assistants. Each executive is assigned exactly 1 assistant, and no assistant is assigned to more than 4 executives. Show that at least 3 assistants are assigned to 3 or more executives.

H ✶ **32.** Let A be a set of six positive integers each of which is less than 13. Show that there must be two distinct subsets of A whose elements when added up give the same sum. (For example, if $A = \{5, 12, 10, 1, 3, 4\}$, then the elements of the subsets $S_1 = \{1, 4, 10\}$ and $S_2 = \{5, 10\}$ both add up to 15.)

H **33.** Let A be a set of six positive integers each of which is less than 15. Show that there must be two distinct subsets of A

whose elements when added up give the same sum. (Thanks to Jonathan Goldstine for this problem.)

34. Let S be a set of ten integers chosen from 1 through 50. Show that the set contains at least two different (but not necessarily disjoint) subsets of four integers that add up to the same number. (For instance, if the ten numbers are $\{3, 8, 9, 18, 24, 34, 35, 41, 44, 50\}$, the subsets can be taken to be $\{8, 24, 34, 35\}$ and $\{9, 18, 24, 50\}$. The numbers in both of these add up to 101.)

H ✶ **35.** Given a set of 52 distinct integers, show that there must be 2 whose sum or difference is divisible by 100.

H ✶ **36.** Show that if 101 integers are chosen from 1 to 200 inclusive, there must be 2 with the property that one is divisible by the other.

✶ 37. a. Suppose $a_1, a_2, \ldots, a_n$ is a sequence of n integers none of which is divisible by n. Show that at least one of the differences $a_i - a_j$ (for $i \neq j$) must be divisible by n.

H **b.** Show that every finite sequence $x_1, x_2, \ldots, x_n$ of n integers has a consecutive subsequence $x_{i+1}, x_{i+2}, \ldots, x_j$ whose sum is divisible by n. (For instance, the sequence 3, 4, 17, 7, 16 has the consecutive subsequence 17, 7, 16 whose sum is divisible by 5.) (From: James E. Schultz and William F. Burger, "An Approach to Problem-Solving Using Equivalence Classes Modulo n," *College Mathematics Journal (15)*, No. 5, 1984, 401–405.)

H ✶ **38.** Observe that the sequence 12, 15, 8, 13, 7, 18, 19, 11, 14, 10 has three increasing subsequences of length four: 12, 15, 18, 19; 12, 13, 18, 19; and 8, 13, 18, 19. It also has one decreasing subsequence of length four: 15, 13, 11, 10. Show that in any sequence of $n^2 + 1$ distinct real numbers, there must be a sequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

✶ 39. What is the largest number of elements that a set of integers from 1 through 100 can have so that no one element in the set is divisible by another? (*Hint:* Imagine writing all the numbers from 1 through 100 in the form $2^k \cdot m$, where $k \geq 0$ and m is odd.)

40. Suppose X and Y are finite sets, X has more elements than Y, and $F: X \to Y$ is a function. By the pigeonhole principle, there exist elements a and b in X such that $a \neq b$ and $F(a) = F(b)$. Write a computer algorithm to find such a pair of elements a and b.