Exercise Set 9.5

1. a. List all 2-combinations for the set $\{x_1, x_2, x_3\}$. Deduce the value of $\binom{3}{2}$.
   b. List all unordered selections of four elements from the set $\{a, b, c, d, e\}$. Deduce the value of $\binom{5}{4}$.

2. a. List all 3-combinations for the set $\{x_1, x_2, x_3, x_4, x_5\}$. Deduce the value of $\binom{5}{3}$.
   b. List all unordered selections of two elements from the set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$. Deduce the value of $\binom{6}{2}$.

3. Write an equation relating $P(7, 2)$ and $\binom{7}{2}$.

4. Write an equation relating $P(8, 3)$ and $\binom{8}{3}$.

5. Use Theorem 9.5.1 to compute each of the following.
   a. $\binom{6}{0}$ b. $\binom{6}{1}$ c. $\binom{6}{2}$
   d. $\binom{6}{3}$ e. $\binom{6}{4}$ f. $\binom{6}{5}$ g. $\binom{6}{6}$

6. A student council consists of 15 students.
   a. In how many ways can a committee of six be selected from the membership of the council?
   b. Two council members have the same major and are not permitted to serve together on a committee. How many ways can a committee of six be selected from the membership of the council?
   c. Two council members always insist on serving on committees together. If they can't serve together, they won't serve at all. How many ways can a committee of six be selected from the council membership?
   d. Suppose the council contains eight men and seven women.
      (i) How many committees of six contain three men and three women?
      (ii) How many committees of six contain at least one woman?
   e. Suppose the council consists of three freshmen, four sophomores, three juniors, and five seniors. How many committees of eight contain two representatives from each class?

7. A computer programming team has 13 members.
   a. How many ways can a group of seven be chosen to work on a project?
   b. Suppose seven team members are women and six are men.
      (i) How many groups of seven can be chosen that contain four women and three men?

(ii) How many groups of seven can be chosen that contain at least one man?

(iii) How many groups of seven can be chosen that contain at most three women?

c. Suppose two team members refuse to work together on projects. How many groups of seven can be chosen to work on a project?

d. Suppose two team members insist on either working together or not at all on projects. How many groups of seven can be chosen to work on a project?

H **8.** An instructor gives an exam with fourteen questions. Students are allowed to choose any ten to answer.

a. How many different choices of ten questions are there?

b. Suppose six questions require proof and eight do not.

(i) How many groups of ten questions contain four that require proof and six that do not?

(ii) How many groups of ten questions contain at least one that requires proof?

(iii) How many groups of ten questions contain at most three that require proof?

c. Suppose the exam instructions specify that at most one of questions 1 and 2 may be included among the ten. How many different choices of ten questions are there?

d. Suppose the exam instructions specify that either both questions 1 and 2 are to be included among the ten or neither is to be included. How many different choices of ten questions are there?

9. A club is considering changing its bylaws. In an initial straw vote on the issue, 24 of the 40 members of the club favored the change and 16 did not. A committee of six is to be chosen from the 40 club members to devote further study to the issue.

a. How many committees of six can be formed from the club membership?

b. How many of the committees will contain at least three club members who, in the preliminary survey, favored the change in the bylaws?

10. Two new drugs are to be tested using a group of 60 laboratory mice, each tagged with a number for identification purposes. Drug A is to be given to 22 mice, drug B is to be given to another 22 mice, and the remaining 16 mice are to be used as controls. How many ways can the assignment of treatments to mice be made? (A single assignment involves specifying the treatment for each mouse—whether drug A, drug B, or no drug.)

✶ 11. Refer to Example 9.5.8. For each poker holding below, (1) find the number of five-card poker hands with that holding; (2) find the probability that a randomly chosen set of five cards has that holding.

a. royal flush b. straight flush **c.** four of a kind

d. full house e. flush **f.** straight

g. three of a kind h. one pair

i. neither a repeated denomination nor five of the same suit nor five adjacent denominations

12. How many pairs of two distinct integers chosen from the set $\{1, 2, 3, \ldots, 101\}$ have a sum that is even?

13. A coin is tossed ten times. In each case the outcome H (for heads) or T (for tails) is recorded. (One possible outcome of the ten tossings is denoted $THHTTTHTTH$.)

a. What is the total number of possible outcomes of the coin-tossing experiment?

b. In how many of the possible outcomes are exactly five heads obtained?

c. In how many of the possible outcomes are at least eight heads obtained?

d. In how many of the possible outcomes is at least one head obtained?

e. In how many of the possible outcomes is at most one head obtained?

14. a. How many 16-bit strings contain exactly seven 1's?

b. How many 16-bit strings contain at least thirteen 1's?

c. How many 16-bit strings contain at least one 1?

d. How many 16-bit strings contain at most one 1?

15. a. How many even integers are in the set

$$\{1, 2, 3, \ldots, 100\}?$$

b. How many odd integers are in the set

$$\{1, 2, 3, \ldots, 100\}?$$

c. How many ways can two integers be selected from the set $\{1, 2, 3, \ldots, 100\}$ so that their sum is even?

d. How many ways can two integers be selected from the set $\{1, 2, 3, \ldots, 100\}$ so that their sum is odd?

16. Suppose that three computer boards in a production run of forty are defective. A sample of five is to be selected to be checked for defects.

a. How many different samples can be chosen?

b. How many samples will contain at least one defective board?

c. What is the probability that a randomly chosen sample of five contains at least one defective board?

17. Ten points labeled $A, B, C, D, E, F, G, H, I, J$ are arranged in a plane in such a way that no three lie on the same straight line.

a. How many straight lines are determined by the ten points?

b. How many of these straight lines do not pass through point A?

c. How many triangles have three of the ten points as vertices?

d. How many of these triangles do not have A as a vertex?

18. Suppose that you placed the letters in Example 9.5.10 into positions in the following order: first the M, then the I's, then the S's, and then the P's. Show that you would obtain the same answer for the number of distinguishable orderings.

19. a. How many distinguishable ways can the letters of the word *HULLABALOO* be arranged in order?

b. How many distinguishable orderings of the letters of *HULLABALOO* begin with U and end with L?

c. How many distinguishable orderings of the letters of *HULLABALOO* contain the two letters HU next to each other in order?

20. a. How many distinguishable ways can the letters of the word *MILLIMICRON* be arranged in order?

b. How many distinguishable orderings of the letters of *MILLIMICRON* begin with M and end with N?

c. How many distinguishable orderings of the letters of *MILLIMICRON* contain the letters CR next to each other in order and also the letters ON next to each other in order?

21. In Morse code, symbols are represented by variable-length sequences of dots and dashes. (For example, $A = \cdot -$, $1 = \cdot - - - -$, $? = \cdot \cdot - - \cdot \cdot$.) How many different symbols can be represented by sequences of seven or fewer dots and dashes?

22. Each symbol in the Braille code is represented by a rectangular arrangement of six dots, each of which may be raised or flat against a smooth background. For instance, when the word Braille is spelled out, it looks like this:

Given that at least one of the six dots must be raised, how many symbols can be represented in the Braille code?

23. On an 8×8 chessboard, a rook is allowed to move any number of squares either horizontally or vertically. How many different paths can a rook follow from the bottom-left square of the board to the top-right square of the board if all moves are to the right or upward?

24. The number 42 has the prime factorization $2 \cdot 3 \cdot 7$. Thus 42 can be written in four ways as a product of two positive integer factors (without regard to the order of the factors): $1 \cdot 42$, $2 \cdot 21$, $3 \cdot 14$, and $6 \cdot 7$. Answer a–d below without regard to the order of the factors.

a. List the distinct ways the number 210 can be written as a product of two positive integer factors.

b. If $n = p_1 p_2 p_3 p_4$, where the p_i are distinct prime numbers, how many ways can n be written as a product of two positive integer factors?

c. If $n = p_1 p_2 p_3 p_4 p_5$, where the p_i are distinct prime numbers, how many ways can n be written as a product of two positive integer factors?

d. If $n = p_1 p_2 \cdots p_k$, where the p_i are distinct prime numbers, how many ways can n be written as a product of two positive integer factors?

25. **a.** How many one-to-one functions are there from a set with three elements to a set with four elements?

b. How many one-to-one functions are there from a set with three elements to a set with two elements?

c. How many one-to-one functions are there from a set with three elements to a set with three elements?

d. How many one-to-one functions are there from a set with three elements to a set with five elements?

H e. How many one-to-one functions are there from a set with m elements to a set with n elements, where $m \le n$?

26. **a.** How many onto functions are there from a set with three elements to a set with two elements?

b. How many onto functions are there from a set with three elements to a set with five elements?

H **c.** How many onto functions are there from a set with three elements to a set with three elements?

d. How many onto functions are there from a set with four elements to a set with two elements?

e. How many onto functions are there from a set with four elements to a set with three elements?

H ✶ **f.** Let $c_{m,n}$ be the number of onto functions from a set of m elements to a set of n elements, where $m \ge n \ge 1$. Find a formula relating $c_{m,n}$ to $c_{m-1,n}$ and $c_{m-1,n-1}$.

27. Let A be a set with eight elements.

a. How many relations are there on A?

b. How many relations on A are reflexive?

c. How many relations on A are symmetric?

d. How many relations on A are both reflexive and symmetric?

H ✶ **28.** A student council consists of three freshmen, four sophomores, four juniors, and five seniors. How many committees of eight members of the council contain at least one member from each class?

✶ 29. An alternative way to derive Theorem 9.5.1 uses the following *division rule*: Let n and k be integers so that k divides n. If a set consisting of n elements is divided into subsets that each contain k elements, then the number of such subsets is n/k. Explain how Theorem 9.5.1 can be derived using the division rule.

30. Find the error in the following reasoning: "Consider forming a poker hand with two pairs as a five-step process.

Step 1: Choose the denomination of one of the pairs.
Step 2: Choose the two cards of that denomination.
Step 3: Choose the denomination of the other of the pairs.
Step 4: Choose the two cards of that second denomination.
Step 5: Choose the fifth card from the remaining denominations.

There are $\binom{13}{1}$ ways to perform step 1, $\binom{4}{2}$ ways to perform step 2, $\binom{12}{1}$ ways to perform step 3, $\binom{4}{2}$ ways to perform step 4, and $\binom{44}{1}$ ways to perform step 5. Therefore, the total number of five-card poker hands with two pairs is $13 \cdot 6 \cdot 12 \cdot 6 \cdot 44 = 247{,}104$."

✶ **31.** Let P_n be the number of partitions of a set with n elements. Show that

$$P_n = \binom{n-1}{0} P_{n-1} + \binom{n-1}{1} P_{n-2} + \cdots + \binom{n-1}{n-1} P_0$$

for all integers $n \ge 1$.

Exercises 32–38 refer to the sequence of Stirling numbers of the second kind.

32. Find $S_{3,4}$ by exhibiting all the partitions of $\{x_1, x_2, x_3, x_4, x_5\}$ into four subsets.

33. Use the values computed in Example 9.5.12 and the recurrence relation and initial conditions found in Example 9.5.13 to compute $S_{5,2}$.

34. Use the values computed in Example 9.5.12 and the recurrence relation and initial conditions found in Example 9.5.13 to compute $S_{5,3}$.

35. Use the results of exercises 32–34 to find the total number of different partitions of a set with five elements.

36. Use mathematical induction and the recurrence relation found in Example 9.5.13 to prove that for all integers $n \geq 2$, $S_{n,2} = 2^{n-1} - 1$.

37. Use mathematical induction and the recurrence relation found in Example 9.5.13 to prove that for all integers $n \geq 2$, $\sum_{k=2}^{k}(3^{4-k} S_{k,2}) - S_{n+1,3}$.

***H* 38.** If X is a set with n elements and Y is a set with m elements, express the number of onto functions from X and Y using Stirling numbers of the second kind. Justify your answer.