Exercise Set 9.7

In 1–4, use Theorem 9.5.1 to compute the values of the indicated quantities. (Assume n is an integer.)

1. $\binom{n}{0}$, for $n \geq 0$ 2. $\binom{n}{1}$, for $n \geq 1$

3. $\binom{n}{2}$, for $n \geq 2$ 4. $\binom{n}{3}$, for $n \geq 3$

5. Use Theorem 9.5.1 to prove algebraically that $\binom{n}{r} = \binom{n}{n-r}$, for integers n and r with $0 \leq r \leq n$. (This can be done by direct calculation; it is not necessary to use mathematical induction.)

Justify the equations in 6–9 either by deriving them from formulas in Example 9.7.1 or by direct computation from Theorem 9.5.1. Assume m, n, k, and r are integers.

6. $\binom{m+k}{m+k-1} = m+k$, for $m+k \geq 1$

7. $\binom{n+3}{n+1} = \dfrac{(n+3)(n+2)}{2}$, for $n \geq -1$

8. $\binom{k-r}{k-r} = 1$, for $k-r \geq 0$

9. $\binom{2n}{n}$ for $n \geq 0$

10. a. Use Pascal's triangle given in Table 9.7.1 to compute the values of $\binom{6}{2}$, $\binom{6}{3}$, $\binom{6}{4}$, and $\binom{6}{5}$.

b. Use the result of part (a) and Pascal's formula to compute $\binom{7}{3}$, $\binom{7}{4}$, and $\binom{7}{5}$.

c. Complete the row of Pascal's triangle that corresponds to $n = 7$.

11. The row of Pascal's triangle that corresponds to $n = 8$ is as follows:

1 8 28 56 70 56 28 8 1.

What is the row that corresponds to $n = 9$?

12. Use Pascal's formula repeatedly to derive a formula for $\binom{n+3}{r}$ in terms of values of $\binom{n}{k}$ with $k \leq r$. (Assume n and r are integers with $n \geq r \geq 3$.)

13. Use Pascal's formula to prove by mathematical induction that if n is an integer and $n \geq 1$, then

$$\sum_{i=2}^{n+1} \binom{i}{2} = \binom{2}{2} + \binom{3}{2} + \cdots + \binom{n+1}{2} = \binom{n+2}{3}.$$

***H* 14.** Prove that if n is an integer and $n \geq 1$, then

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = 2\binom{n+2}{3}.$$

15. Prove the following generalization of exercise 13: Let r be a fixed nonnegative integer. For all integers n with $n \geq r$,

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}.$$

16. Think of a set with $m+n$ elements as composed of two parts, one with m elements and the other with n elements. Give a combinatorial argument to show that

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0},$$

where m and n are positive integers and r is an integer that is less than or equal to both m and n.

This identity gives rise to many useful additional identities involving the quantities $\binom{n}{k}$. Because Alexander Vandermonde published an influential article about it in 1772, it is generally called the *Vandermonde convolution*. However, it was known at least in the 1300s in China by Chu Shih-chieh.

***H* 17.** Prove that for all integers $n \geq 0$,

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

18. Let m be any nonnegative integer. Use mathematical induction and Pascal's formula to prove that for all integers $n \geq 0$,

$$\binom{m}{0} + \binom{m+1}{1} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}.$$

Use the binomial theorem to expand the expressions in 19–27.

19. $(1+x)^7$ **20.** $(p+q)^6$ **21.** $(1-x)^6$

22. $(u-v)^5$ **23.** $(p-2q)^4$ **24.** $(u^2-3v)^4$

25. $\left(x+\frac{1}{x}\right)^5$ **26.** $\left(\frac{3}{a}-\frac{a}{3}\right)^5$ **27.** $\left(x^2+\frac{1}{x}\right)^5$

28. In Example 9.7.5 it was shown that

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Evaluate $(a+b)^6$ by substituting the expression above into the equation

$$(a+b)^6 = (a+b)(a+b)^5$$

and then multiplying out and combining like terms.

In 29–34, find the coefficient of the given term when the expression is expanded by the binomial theorem.

29. x^6y^3 in $(x+y)^9$

30. x^7 in $(2x+3)^{10}$

31. a^5b^7 in $(a-2b)^{12}$

32. $u^{16}v^4$ in $(u^2-v^2)^{10}$

33. $p^{16}q^7$ in $(3p^2-2q)^{15}$

34. x^9y^{10} in $(2x-3y^2)^{14}$

35. As in the proof of the binomial theorem, transform the summation

$$\sum_{k=0}^{n}\binom{m}{k}a^{m-k}b^{k+1}$$

by making the change of variable $j=k+1$.

Use the binomial theorem to prove each statement in 36–41.

36. For all integers $n \geq 1$,

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=0.$$

(*Hint:* Use the fact that $1+(-1)=0$.)

***H* 37.** For all integers $n \geq 0$,

$$3^n=\binom{n}{0}+2\binom{n}{1}+2^2\binom{n}{2}+\cdots+2^n\binom{n}{n}.$$

38. For all integers $m \geq 0$, $\sum_{i=0}^{m}(-1)^i\binom{m}{i}2^{m-i}=1$.

39. For all integers $n \geq 0$, $\sum_{i=0}^{n}(-1)^i\binom{n}{i}3^{n-i}=2^n$.

40. For all integers $n \geq 0$ and for all nonnegative real numbers x, $1+nx \leq (1+x)^n$.

***H* 41.** For all integers $n \geq 1$,

$$\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{2^2}\binom{n}{2}-\frac{1}{2^3}\binom{n}{3}+\cdots+(-1)^{n-1}\frac{1}{2^{n-1}}\binom{n}{n-1}=\begin{cases}0 & \text{if } n \text{ is even}\\ \dfrac{1}{2^{n-1}} & \text{if } n \text{ is odd}\end{cases}.$$

42. Use mathematical induction to prove that for all integers $n \geq 1$, if S is a set with n elements, then S has the same number of subsets with an even number of elements as with an odd number of elements. Use this fact to give a combinatorial argument to justify the identity of exercise 36.

Express each of the sums in 43–54 in closed form (without using a summation symbol and without using an ellipsis $\cdots$).

43. $\sum_{k=0}^{n}\binom{n}{k}5^k$

44. $\sum_{i=0}^{m}\binom{m}{i}4^i$

45. $\sum_{i=0}^{n}\binom{n}{i}x^i$

46. $\sum_{k=0}^{m}\binom{m}{k}2^{m-k}x^k$

47. $\sum_{j=0}^{2n}(-1)^j\binom{2n}{j}x^j$

48. $\sum_{r=0}^{n}\binom{n}{r}x^{2r}$

49. $\sum_{i=0}^{m}\binom{m}{i}p^{m-i}q^{2i}$

50. $\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^k}$

51. $\sum_{i=0}^{m}(-1)^i\binom{m}{i}\frac{1}{2^i}$

52. $\sum_{k=0}^{n}\binom{n}{k}3^{2n-2k}2^{2k}$

53. $\sum_{i=0}^{n}(-1)^i\binom{n}{i}5^{n-i}2^i$

54. $\sum_{k=0}^{n}(-1)^k\binom{n}{k}3^{2n-2k}2^{2k}$

✶ 55. (For students who have studied calculus)

a. Explain how the equation below follows from the binomial theorem:

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k.$$

b. Write the formula obtained by taking the derivative of both sides of the equation in part (a) with respect to x.

c. Use the result of part (b) to derive the formulas below.

(i) $2^{n-1}=\frac{1}{n}\left[\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n}\right]$

(ii) $\sum_{k=1}^{n}k\binom{n}{k}(-1)^k=0$

d. Express $\sum_{k=1}^{n}k\binom{n}{k}3^k$ in closed form (without using a summation sign or ellipsis).