Exercise Set 9.9

1. Suppose $P(A \mid B) = 1/2$ and $P(A \cap B) = 1/6$. What is $P(B)$?

2. Suppose $P(X \mid Y) = 1/3$ and $P(Y) = 1/4$. What is $P(X \cap Y)$?

H 3. The instructor of a discrete mathematics class gave two tests. Twenty-five percent of the students received an A on the first test and 15% of the students received A's on both tests. What percent of the students who received A's on the first test also received A's on the second test?

4. a. Prove that if A and B are any events in a sample space S, with $P(B) \neq 0$, then $P(A^c \mid B) = 1 - P(A \mid B)$.
   b. Explain how this result justifies the following statements: (1) If the probability of a false positive on a test for a condition is 4%, then there is a 96% probability that a person who does not have the condition will have a negative test result. (2) If the probability of a false negative on a test for a condition is 1%, then there is a 99% probability that a person who does have the condition will test positive for it.

H 5. Suppose that A and B are events in a sample space S and that $P(A)$, $P(B)$, and $P(A \mid B)$ are known. Derive a formula for $P(A \mid B^c)$.

6. An urn contains 25 red balls and 15 blue balls. Two are chosen at random, one after the other, without replacement.
   a. Use a tree diagram to help calculate the following probabilities: the probability that both balls are red, the probability that the first ball is red and the second is not, the probability that the first ball is not red and the second is red, the probability that neither ball is red.
   b. What is the probability that the second ball is red?
   c. What is the probability that at least one of the balls is red?

7. Redo exercise 6 assuming that the urn contains 30 red balls and 40 blue balls.

8. A pool of 10 semifinalists for a job consists of 7 men and 3 women. Because all are considered equally qualified, the names of two of the semifinalists are drawn, one after the other, at random, to become finalists for the job.
   a. What is the probability that both finalists are women?
   b. What is the probability that both finalists are men?
   H c. What is the probability that one finalist is a woman and the other is a man?

H 9. Prove Bayes' Theorem for $n = 2$. That is, prove that if a sample space S is a union of mutually disjoint events B_1 and B_2, if A is an event in S with $P(A) \neq 0$, and if $k = 1$ or $k = 2$, then

$$P(B_k \mid A) = \frac{P(A \mid B_k)P(B_k)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)}.$$

10. Prove the full version of Bayes' Theorem.

11. One urn contains 12 blue balls and 7 white balls, and a second urn contains 8 blue balls and 19 white balls. An urn is selected at random, and a ball is chosen from the urn.
    a. What is the probability that the chosen ball is blue?
    b. If the chosen ball is blue, what is the probability that it came from the first urn?

12. Redo exercise 11 assuming that the first urn contains 4 blue balls and 16 white balls and the second urn contains 10 blue balls and 9 white balls.

H 13. One urn contains 10 red balls and 25 green balls, and a second urn contains 22 red balls and 15 green balls. A ball is chosen as follows: First an urn is selected by tossing a loaded coin with probability 0.4 of landing heads up and probability 0.6 of landing tails up. If the coin lands heads up, the first urn is chosen; otherwise, the second urn is chosen. Then a ball is picked at random from the chosen urn.
    a. What is the probability that the chosen ball is green?
    b. If the chosen ball is green, what is the probability that it was picked from the first urn?

14. A drug-screening test is used in a large population of people of whom 4% actually use drugs. Suppose that the false positive rate is 3% and the false negative rate is 2%. Thus a person who uses drugs tests positive for them 98% of the time, and a person who does not use drugs tests negative for them 987% of the time.
    a. What is the probability that a randomly chosen person who tests positive for drugs actually uses drugs?
    b. What is the probability that a randomly chosen person who tests negative for drugs does not use drugs?

15. Two different factories both produce a certain automobile part. The probability that a component from the first factory is defective is 2%, and the probability that a component from the second factory is defective is 5%. In a supply of 180 of the parts, 100 were obtained from the first factory and 80 from the second factory.
    a. What is the probability that a part chosen at random from the 180 is from the first factory?
    b. What is the probability that a part chosen at random from the 180 is from the second factory?
    c. What is the probability that a part chosen at random from the 180 is defective?
    d. If the chosen part is defective, what is the probability that it came from the first factory?

H 16. Three different suppliers—X, Y, and Z—provide produce for a grocery store. Twelve percent of produce from X is superior grade, 8% of produce from Y is superior grade and 15% of produce from Z is superior grade. The store obtains 20% of its produce from X, 45% from Y, and 35% from Z.
    a. If a piece of produce is purchased, what is the probability that it is superior grade?
    b. If a piece of produce in the store is superior grade, what is the probability that it is from X?

17. Prove that if A and B are events in a sample space S with the property that $P(A \mid B) = P(A)$ and $P(A) \neq 0$, then $P(B \mid A) = P(B)$.

18. Prove that if $P(A \cap B) = P(A) \cdot P(B)$, $P(A) \neq 0$, and $P(B) \neq 0$, then $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$.

19. A pair of fair dice, one blue and the other gray, are rolled. Let A be the event that the number face up on the blue die is 2, and let B be the event that the number face up on the gray die is 4 or 5. Show that $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$.

20. Suppose a fair coin is tossed three times. Let A be the event that a head appears on the first toss, and let B be the event that an even number of heads is obtained. Show that $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$.

21. If A and B are events in a sample space S and $A \cap B = \emptyset$, what must be true in order for A and B to be independent? Explain.

22. Prove that if A and B are independent events in a sample space S, then A^c and B are also independent, and so are A^c and B^c.

23. A student taking a multiple-choice exam does not know the answers to two questions. All have five choices for the answer. For one of the two questions, the student can eliminate two answer choices as incorrect but has no idea about the other answer choices. For the other question, the student has no clue about the correct answer at all. Assume that whether the student chooses the correct answer on one of the questions does not affect whether the student chooses the correct answer on the other question.
    a. What is the probability that the student will answer both questions correctly?
    b. What is the probability that the student will answer exactly one of the questions correctly?
    c. What is the probability that the student will answer neither question correctly?

24. A company uses two proofreaders X and Y to check a certain manuscript. X misses 12% of typographical errors and Y misses 15%. Assume that the proofreaders work independently.
    a. What is the probability that a randomly chosen typographical error will be missed by both proofreaders?
    b. If the manuscript contains 1,000 typographical errors, what number can be expected to be missed?

25. A coin is loaded so that the probability of heads is 0.7 and the probability of tails is 0.3. Suppose that the coin is tossed twice and that the results of the tosses are independent.
    a. What is the probability of obtaining exactly two heads?
    b. What is the probability of obtaining exactly one head?
    c. What is the probability of obtaining no heads?
    d. What is the probability of obtaining at least one head?

★26. Describe a sample space and events A, B, and C, where $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ but A, B, and C are not pairwise independent.

H 27. The example used to introduce conditional probability described a family with two children each of whom was equally likely to be a boy or a girl. The example showed that if it is known that one child is a boy, the probability that the other child is a boy is 1/3. Now imagine the same kind of family—two children each of whom is equally likely to be a boy or a girl. Suppose you meet one of the children and see that it is a boy. What is the probability that the other child is a boy? Explain. (Be careful. The answer may surprise you.)

28. A coin is loaded so that the probability of heads is 0.7 and the probability of tails is 0.3. Suppose that the coin is tossed ten times and that the results of the tosses are mutually independent.
   a. What is the probability of obtaining exactly seven heads?
   b. What is the probability of obtaining exactly ten heads?
   c. What is the probability of obtaining no heads?
   d. What is the probability of obtaining at least one head?

29. Suppose that ten items are chosen at random from a large batch delivered to a company. The manufacturer claims that just 3% of the items in the batch are defective. Assume that the batch is large enough so that even though the selection is made without replacement, the number 0.03 can be used to approximate the probability that any one of the ten items is defective. In addition, assume that because the items are chosen at random, the outcomes of the choices are mutually independent. Finally, assume that the manufacturer's claim is correct.
   a. What is the probability that none of the ten is defective?
   b. What is the probability that at least one of the ten is defective?
   c. What is the probability that exactly four of the ten are defective?
   d. What is the probability that at most two of the ten are defective?

30. Suppose the probability of a false positive result on a mammogram is 4% and that radiologists' interpretations of mammograms are mutually independent in the sense that whether or not a radiologist finds a positive result on one mammogram does not influence whether or not the radiologist finds a positive result on another mammogram. Assume that a woman has a mammogram every year for ten years.
   a. What is the probability that she will have no false positive results during that time?
   b. What is the probability that she will have at least one false positive result during that time?
   c. What is the probability that she will have exactly two false positive results during that time?
   d. Suppose that the probability of a false negative result on a mammogram is 2%, and assume that the probability that a randomly chosen woman has breast cancer is 0.0002.
      (i) If a woman has a positive test result one year, what is the probability that she actually has breast cancer?
      (ii) If a woman has a negative test result one year, what is the probability that she actually has breast cancer?

31. Empirical data indicate that approximately 103 out of every 200 children born are male. Hence the probability of a newborn being male is about 51.5%. Suppose that a family has six children, and suppose that the genders of all the children are mutually independent.
   H a. What is the probability that none of the children is male?
   b. What is the probability that at least one of the children is male?
   c. What is the probability that exactly five of the children are male?

32. A person takes a multiple-choice exam in which each question has four possible answers. Suppose that the person has no idea about the answers to three of the questions and simply chooses randomly for each one.
   a. What is the probability that the person will answer all three questions correctly?
   b. What is the probability that the person will answer exactly two questions correctly?
   c. What is the probability that the person will answer exactly one question correctly?
   d. What is the probability that the person will answer no questions correctly?
   e. Suppose that the person gets one point of credit for each correct answer and that 1/3 point is deducted for each incorrect answer. What is the expected value of the person's score for the three questions?

33. In exercise 23 of Section 9.8, let C_k be the event that the gambler has k dollars, wins the next roll of the die, and is eventually ruined, let D_k be the event that the gambler has k dollars, loses the next roll of the die, and is eventually ruined, and let P_n be the probability that the gambler is eventually ruined. Use the probability axioms and the definition of conditional probability to derive the equation $P_{k-1} = \frac{1}{6}P_k + \frac{5}{6}P_{k-2}$.